

Hypergeometric functions and algebraic curves $y^e = x^d + ax + b$

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Abstract. Let q be a prime power and \mathbb{F}_q be a finite field with q elements. Let e and d be positive integers. In this paper, for $d \geq 2$ and $q \equiv 1 \pmod{ed(d-1)}$, we calculate the number of points on an algebraic curve $E_{e,d} : y^e = x^d + ax + b$ over a finite field \mathbb{F}_q in terms of ${}_dF_{d-1}$ Gaussian hypergeometric series with multiplicative characters of orders d and $e(d-1)$, and in terms of ${}_{d-1}F_{d-2}$ Gaussian hypergeometric series with multiplicative characters of orders $ed(d-1)$ and $e(d-1)$. This helps us to express the trace of Frobenius endomorphism of an algebraic curve $E_{e,d}$ over a finite field \mathbb{F}_q in terms of above hypergeometric series. As applications, we obtain some transformations and special values of ${}_2F_1$ hypergeometric series.

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1. Introduction

In the 19th century Gauss introduced classical hypergeometric series. Since then, many mathematicians such as Kummer, Ramanujan, Beukers, Stiller and others studied classical hypergeometric series extensively and found many interesting connections between classical hypergeometric series and different mathematical objects. In 1980's, Greene [7] introduced hypergeometric functions (or Gaussian hypergeometric series) over finite fields analogous to classical hypergeometric series as finite character sums over a finite field. It is found that these functions satisfy many summation and transformation formulas analogous to classical hypergeometric series. In a series of papers, many interesting relations have been established between special values of these hypergeometric functions and the number of points on certain algebraic curves over finite fields (see, for examples, [1–6,9,10,12,14]).

Fuselier [6] and Lennon [12] found formulas for the trace of Frobenius endomorphism of a certain family of elliptic curves in terms of Gaussian hypergeometric series containing characters of order 12. In [4], Barman and Kalita found the number of solutions of the polynomial equation $x^d + ax + b = 0$ over a finite field \mathbb{F}_q in terms of special values of Gaussian hypergeometric series with characters of orders d and $d-1$ under the condition that $q \equiv 1 \pmod{d(d-1)}$ and $d \geq 2$. The same authors, in [5], expressed the number of \mathbb{F}_q -points on a hyperelliptic curve in terms of special values of Gaussian hypergeometric series.

Let e and d be positive integers and $E_{e,d} : y^e = x^d + ax + b$ be an algebraic curve over a finite field \mathbb{F}_q . Throughout this paper, we assume that $a, b \neq 0$. Let $N_{e,d}$ denotes the number of points on the algebraic curve $E_{e,d}$ over \mathbb{F}_q excluding points at infinity and $a_q(E_{e,d})$ denotes the trace of Frobenius of the algebraic curve $E_{e,d}$. In this paper, for $d \geq 2$ and $q \equiv 1 \pmod{ed(d-1)}$, we express $N_{e,d}$ and $a_q(E_{e,d})$ in terms of ${}_dF_{d-1}$ and ${}_{d-1}F_{d-2}$ Gaussian hypergeometric series containing multiplicative characters of orders d , $e(d-1)$ and $ed(d-1)$. We deduce the result of Lennon [12] on the trace of Frobenius of an elliptic curve from our main results. In [14], Ono obtained special values of hypergeometric functions containing quadratic and trivial characters. Only a few such values are known for heigher order characters. In the last section, we derive some interesting special values of ${}_2F_1$ hypergeometric function containing multiplicative characters of order 12.

2. Preliminaries

Let \mathbb{F}_q be a finite field with q elements, where $q = p^n$, p is a prime number and n is a positive integer. Note that $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ is a cyclic multiplicative group of order $q - 1$. A multiplicative character $\chi: \mathbb{F}_q^* \rightarrow \mathbb{C}^*$ is a group homomorphism. Throughout, we reserve the notations ε and ϕ for the trivial and the quadratic characters, respectively. Thus, for $x \in \mathbb{F}_q^*$

$$\phi(x) = \left(\frac{x}{q}\right) = \begin{cases} 1, & \text{if } x \text{ is a square of some element in } \mathbb{F}_q^*, \\ -1, & \text{if } x \text{ is not a square of any element in } \mathbb{F}_q^* \end{cases} \tag{2.1}$$

is the Legendre symbol. The following theorem gives the structure of multiplicative characters on \mathbb{F}_q^* .

Theorem 2.1 ([13]). *Let g be a generator of the multiplicative group \mathbb{F}_q^* . For $j = 0, 1, 2, \dots, q - 2$, the functions*

$$\chi_j(g^k) = e^{\frac{2\pi ijk}{q-1}}, \quad \text{for } k = 0, 1, 2, \dots, q - 2,$$

define multiplicative characters on \mathbb{F}_q^* .

The set $\widehat{\mathbb{F}_q^*}$ of all multiplicative characters on \mathbb{F}_q^* is a cyclic group under multiplication of characters. One extends the domain of a multiplicative character χ on \mathbb{F}_q^* to \mathbb{F}_q by defining $\chi(0) = 0$.

Define the additive character $\theta: \mathbb{F}_q \rightarrow \mathbb{C}^*$ by $\theta(\alpha) = \zeta^{tr(\alpha)}$, where $\zeta = e^{\frac{2\pi i}{p}}$ and $tr: \mathbb{F}_q \rightarrow \mathbb{F}_p$ is the trace map given by

$$tr(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \dots + \alpha^{p^{n-1}}.$$

Throughout this paper, by capital letters A, B, C, \dots and Greek letters χ, ψ, \dots , we will denote multiplicative characters. Let δ denote both the function on \mathbb{F}_q and the function on $\widehat{\mathbb{F}_q^*}$:

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{if } x \neq 0, \end{cases} \tag{2.2}$$

and

$$\delta(A) = \begin{cases} 1 & \text{if } A \text{ is trivial character,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Define $\overline{\chi}$ by $\chi\overline{\chi} = \varepsilon$. We write \sum_x to denote the sum over all x in \mathbb{F}_q and \sum_χ to denote the sum over all characters of \mathbb{F}_q .

We recall the definitions of the Jacobi sum

$$J(A, B) = \sum_x A(x)B(1-x) \tag{2.4}$$

and the Gauss sum

$$G(A) =: \sum_x A(x)\zeta^{tr(x)}. \tag{2.5}$$

Definition 2.2 ([13]). *For characters A and B of \mathbb{F}_q , the binomial coefficient $\binom{A}{B}$ is defined as*

$$\binom{A}{B} = \frac{B(-1)}{q} J(A, \overline{B}).$$

In terms of the binomial coefficients, $A(1+x)$ can be written as

$$A(1+x) = \delta(x) + \frac{q}{q-1} \sum_\chi \binom{A}{\chi} \chi(x). \tag{2.6}$$

Some useful properties of the binomial coefficients which follow easily from properties of the Jacobi sums are

$$\binom{A}{B} = \binom{A}{A\bar{B}}, \quad (2.7)$$

$$\binom{A}{B} = \binom{B\bar{A}}{B} B(-1) \quad (2.8)$$

and

$$\binom{A}{B} = \binom{\bar{B}}{\bar{A}} AB(-1). \quad (2.9)$$

Let T be a fixed generator of $\widehat{\mathbb{F}}_q^*$ and $G_m = G(T^m)$. The following lemma gives the beautiful properties of the Gauss sum.

Lemma 2.3 ([6]). *If T^m is not trivial for $m \in \mathbb{N}$, then we have*

$$G_m G_{-m} = q T^m(-1).$$

The following lemmas give the nice relationship between the Gauss sum and the Jacobi sum.

Lemma 2.4 ([12]). *If T^{m-n} is not trivial for $m, n \in \mathbb{N}$, then we have*

$$\begin{aligned} G_m G_{-n} &= q \binom{T^m}{T^n} G_{m-n} T^n(-1) \\ &= J(T^m, T^{-n}) G_{m-n}. \end{aligned}$$

Lemma 2.5 ([13]). *If $T^{k_1}, T^{k_2}, \dots, T^{k_n}$ are nontrivial multiplicative characters of \mathbb{F}_q and $T^{k_1+k_2+\dots+k_n}$ is nontrivial, then*

$$J(T^{k_1}, T^{k_2}, \dots, T^{k_n}) = \frac{G_{k_1} G_{k_2} \cdots G_{k_n}}{G_{k_1+k_2+\dots+k_n}}.$$

Lemma 2.6 ([6]). *If T is a fixed generator of $\widehat{\mathbb{F}}_q^*$ and $n \in \mathbb{N}$, then orthogonality relations for multiplicative characters are given by*

$$(1) \sum_{x \in \mathbb{F}_q^*} T^n(x) = \begin{cases} q-1 & \text{if } T^n = \epsilon; \\ 0 & \text{if } T^n \neq \epsilon. \end{cases}$$

$$(2) \sum_{n=0}^{q-2} T^n(x) = \begin{cases} q-1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases}$$

The following lemma gives the relation between additive characters and the Gauss sums.

Lemma 2.7 ([6]). *Let θ be an additive character and $\alpha \in \mathbb{F}_q^*$. We have*

$$\theta(\alpha) = \frac{1}{q-1} \sum_{m=0}^{q-2} G_{-m} T^m(\alpha).$$

Theorem 2.8 (Davenport-Hasse Relation [11]). *Let $m \in \mathbb{Z}^+$ and $q \equiv 1 \pmod{m}$. For multiplicative characters $\chi, \psi \in \widehat{\mathbb{F}}_q^*$, we have*

$$\prod_{\chi^m=1} G(\chi\psi) = -G(\psi^m)\psi(m^{-m}) \prod_{\chi^m=1} G(\chi).$$

We have the following two special cases of the above theorem.

Corollary 2.9 ([4]). Let d be a positive integer, $l \in \mathbb{Z}$, $q \equiv 1 \pmod{d}$ and $t \in \{1, -1\}$. If $d > 1$ is an odd integer, then

$$G_l G_{l+t \frac{q-1}{d}} G_{l+t \frac{2(q-1)}{d}} \cdots G_{l+t \frac{(d-1)(q-1)}{d}} = q^{\frac{d-1}{2}} T^{\frac{(d-1)(d+1)(q-1)}{8d}} (-1)^l T^{-l} (d^d) G_{ld}.$$

If d is an even integer, then

$$G_l G_{l+t \frac{q-1}{d}} G_{l+t \frac{2(q-1)}{d}} \cdots G_{l+t \frac{(d-1)(q-1)}{d}} = q^{\frac{d-2}{2}} G_{\frac{q-1}{2}} T^{\frac{(d-2)(q-1)}{8}} (-1)^l T^{-l} (d^d) G_{ld}.$$

The following lemma gives the values of the Gauss sum at the trivial and the quadratic characters.

Lemma 2.10 ([6]). We have

- (1) $G(\epsilon) = G_0 = -1$,
- (2) $G(\phi) = G_{\frac{q-1}{2}} = \begin{cases} \sqrt{q} & \text{if } q \equiv 1 \pmod{4}; \\ i\sqrt{q} & \text{if } q \equiv 3 \pmod{4}. \end{cases}$

Theorem 2.11 ([8]). Let θ be an additive character and $x, y, z \in \mathbb{F}_q$. Then we have

$$\sum_{z \in \mathbb{F}_q} \theta(z(x - y)) = q\delta(x, y),$$

where $\delta(x, y) = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}$

Definition 2.12 ([13]). If A_0, A_1, \dots, A_n and B_1, B_2, \dots, B_n are characters of \mathbb{F}_q and $x \in \mathbb{F}_q$, then the Gaussian hypergeometric series ${}_{n+1}F_n$ over \mathbb{F}_q is defined as

$${}_{n+1}F_n \left(\begin{matrix} A_0, A_1, \dots, A_n \\ B_1, \dots, B_n \end{matrix} \middle| x \right) = \frac{q}{q-1} \sum_{\chi} \binom{A_0 \chi}{\chi} \binom{A_1 \chi}{B_1 \chi} \cdots \binom{A_n \chi}{B_n \chi} \chi(x).$$

Theorem 2.13 ([7]). For characters A, B, C on \mathbb{F}_q and $x \in \mathbb{F}_q$,

$$(1) {}_2F_1 \left(\begin{matrix} A, B \\ C \end{matrix} \middle| x \right) = A(-1) {}_2F_1 \left(\begin{matrix} A, B \\ A\overline{B}C \end{matrix} \middle| 1-x \right) + A(-1) \binom{B}{A\overline{C}} \delta(1-x) - \binom{B}{C} \delta(x),$$

$$(2) {}_2F_1 \left(\begin{matrix} A, B \\ C \end{matrix} \middle| x \right) = C(-1) \overline{A}(1-x) {}_2F_1 \left(\begin{matrix} A, C\overline{B} \\ C \end{matrix} \middle| \frac{x}{x-1} \right) + A(-1) \binom{B}{A\overline{C}} \delta(1-x).$$

3. Main Theorems

In this section, we state and prove the main theorems of this paper.

Theorem 3.1. Let e and d be positive integers and let $N_{e,d}$ denotes the number of \mathbb{F}_q points on $E_{e,d} : y^e = x^d + ax + b$ excluding points at infinity. If $d \geq 2$ and is an even integer, and $q \equiv 1 \pmod{ed(d-1)}$, then

$$N_{e,d} = q + \sum_{i=1}^{e-1} T^{-\frac{i(q-1)}{e}}(b) + \frac{1}{T^{\frac{(d-2)(2d-1)(q-1)}{8(d-1)}}(-1)} \sum_{i=1}^{e-1} M_i T^{\frac{i(q-1)}{e}} \left(\frac{d-1}{b} \right) \\ \times {}_dF_{d-1} \left(\begin{matrix} \phi, & \epsilon, & \chi, & \chi^2, \dots, \chi^{\frac{d-2}{2}}, & \chi^{\frac{d+2}{2}}, \dots, \chi^{d-1} \\ \psi^{\left(\frac{d}{2}e-i\right)}, & \psi^{e-i}, & \psi^{2e-i}, \dots, \psi^{\frac{d-2}{2}e-i}, & \psi^{\frac{d+2}{2}e-i}, \dots, \psi^{(d-1)e-i} \end{matrix} \middle| \alpha \right),$$

where $\alpha = \frac{d}{a} \left(\frac{bd}{a(d-1)} \right)^{d-1}$, χ and ψ are characters of orders d and $e(d-1)$ respectively and

$$M_i = G_{-\frac{i(q-1)}{e}} G_{-\frac{\left(\frac{d}{2}e-i\right)(q-1)}{e(d-1)}} G_{\frac{(id-e)(q-1)}{ed(d-1)}} G_{\frac{(id-2e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-\left(\frac{d}{2}-1\right)e)(q-1)}{ed(d-1)}} \\ \times G_{\frac{id-\left(\frac{d}{2}+1\right)e(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}}, \quad i = 1, 2, \dots, (e-1).$$

Remark 3.2. In the above theorem, by using Lemmas 2.4 and 2.5, we can simplify the expression of M_i . If $e = 2$, then $M_1 = q^{\frac{d}{2}} T^{\frac{q-1}{2}}(-1)$, and if $e \neq 2$, then

$$M_i = q^2 \frac{G_{\frac{(2i-e)(q-1)}{2e(d-1)}}}{G_{\frac{(id-\frac{d}{2}e)(q-1)}{ed(d-1)}}} \left(T^{\frac{(2i-e)(q-1)}{2e}} \right) \left(T^{\frac{q-1}{2}} \right) T^{\frac{(2i(d-2)+ed)(q-1)}{2e(d-1)}}(-1) \\ \times J \left(T^{\frac{(id-e)(q-1)}{ed(d-1)}}, T^{\frac{(id-2e)(q-1)}{ed(d-1)}}, \dots, T^{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}} \right), \quad i = 1, 2, \dots, (e-1).$$

Theorem 3.3. Let e and d be positive integers and let $N_{e,d}$ denotes the number of \mathbb{F}_q points on $E_{e,d} : y^e = x^d + ax + b$ excluding points at infinity. If $d \geq 3$ is an odd integer, and $q \equiv 1 \pmod{ed(d-1)}$, then

$$N_{e,d} = q + \sum_{i=1}^{e-1} T^{-\frac{i(q-1)}{e}}(b) - \frac{T^{\frac{(3d-1)(q-1)}{8d}}(-1)}{q^{(d-1)} G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} N_i G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}}(-1) T^{\frac{(e-i)(q-1)}{e}} \left(\frac{b}{d-1} \right) \\ + \frac{T^{\frac{(4d^2+3d-1)(q-1)}{8d}}(-1)}{G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1} \right) T^{\frac{(e-i)(q-1)}{e(d-1)}} \left(-\frac{1}{\alpha} \right) M_i \\ \times {}_{d-1}F_{d-2} \left(\begin{matrix} \eta^{id-e}, & \eta^{de+id-2e}, \dots, \eta^{\frac{ed^2-4ed+2id+e}{2}}, & \eta^{\frac{ed^2-2ed+2id-e}{2}}, \dots, \eta^{ed^2-3ed+id+e} \\ \psi^e, \dots, \psi^{\frac{d-3}{2}e}, & \psi^{\frac{d-1}{2}e}, \dots, \psi^{(d-2)e} \end{matrix} \middle| -\alpha \right),$$

where $\alpha = \frac{d}{a} \left(\frac{bd}{a(d-1)} \right)^{d-1}$, η and ψ are characters of orders $ed(d-1)$ and $e(d-1)$ respectively,

$$M_i = G_{\frac{(id-e)(q-1)}{ed(d-1)}} G_{\frac{(id-2e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-\frac{d-1}{2}e)(q-1)}{ed(d-1)}} G_{\frac{(id-\frac{d+1}{2}e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}}$$

and

$$N_i = \left\{ G_{\frac{q-1}{d}} G_{-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{\frac{2(q-1)}{d}} G_{-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \cdots \left\{ G_{\frac{(d-1)(q-1)}{2d}} G_{-\frac{\left(\frac{d-1}{2}e-i\right)(q-1)}{e(d-1)}} \right\} \\ \times \left\{ G_{\frac{(d+1)(q-1)}{2d}} G_{-\frac{\left(\frac{d+1}{2}e-i\right)(q-1)}{e(d-1)}} \right\} \cdots \left\{ G_{\frac{(d-1)(q-1)}{d}} G_{-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right\}, \quad i = 1, 2, \dots, (e-1).$$

Remark 3.4. In the above theorem, by using Lemmas 2.3, 2.4 and 2.5, we can simplify the expressions of M_i and N_i . If $e = 2$, then $M_1 = q^{\frac{d-1}{2}} T^{-\frac{(d-1)(q-1)}{8d}} (-1)$, and $N_1 = q^{d-1} T^{-\frac{(d-1)(q-1)}{8d}} (-1)$. If $e \neq 2$, then

$$M_i = G_{\frac{(2i-e)(q-1)}{2e}} J \left(T^{\frac{(id-e)(q-1)}{ed(d-1)}}, T^{\frac{(id-2e)(q-1)}{ed(d-1)}}, \dots, T^{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}} \right), \quad i = 1, 2, \dots, (e - 1)$$

and

$$N_i = q^{\frac{d-1}{2}} T^{\frac{(d^2-1)(q-1)}{8d}} (-1) G_{-\frac{(ed-2i)(q-1)}{2e}} J \left(T^{-\frac{(e-i)(q-1)}{e(d-1)}}, T^{-\frac{(2e-i)(q-1)}{e(d-1)}}, \dots, T^{-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right), \quad i = 1, 2, \dots, (e - 1).$$

Proof of the Theorem 3.1. Let $P(x, y) = x^d + ax + b - y^e$. Then $N_{e,d} = \#\{(x, y) \in \mathbb{F}_q^2 : P(x, y) = 0\}$. Now using the elementary identity,

$$\sum_{z \in \mathbb{F}_q} \theta(zP(x, y)) = \begin{cases} q & \text{if } P(x, y) = 0; \\ 0 & \text{if } P(x, y) \neq 0, \end{cases}$$

we obtain that

$$\begin{aligned} qN_{e,d} &= \sum_{x,y,z \in \mathbb{F}_q} \theta(zP(x, y)) = \sum_{x,y \in \mathbb{F}_q} \theta(0P(x, y)) + \sum_{z \in \mathbb{F}_q^*} \theta(zP(0, 0)) + \sum_{y,z \in \mathbb{F}_q^*} \theta(zP(0, y)) \\ &+ \sum_{x,z \in \mathbb{F}_q^*} \theta(zP(x, 0)) + \sum_{x,y,z \in \mathbb{F}_q^*} \theta(zP(x, y)) \\ &= q^2 + \sum_{z \in \mathbb{F}_q^*} \theta(bz) + \sum_{y,z \in \mathbb{F}_q^*} \theta(bz)\theta(-zy^e) + \sum_{x,z \in \mathbb{F}_q^*} \theta(bz)\theta(zx^d)\theta(azx) \\ &+ \sum_{x,y,z \in \mathbb{F}_q^*} \theta(bz)\theta(zx^d)\theta(azx)\theta(-zy^e). \end{aligned}$$

This we can write as

$$qN_{e,d} := q^2 + A + B + C + D, \tag{3.1}$$

where

$$\begin{aligned} A &= \sum_{z \in \mathbb{F}_q^*} \theta(bz), \quad B = \sum_{y,z \in \mathbb{F}_q^*} \theta(bz)\theta(-zy^e), \\ C &= \sum_{x,z \in \mathbb{F}_q^*} \theta(bz)\theta(zx^d)\theta(azx) \quad \text{and} \quad D = \sum_{x,y,z \in \mathbb{F}_q^*} \theta(bz)\theta(zx^d)\theta(azx)\theta(-zy^e). \end{aligned}$$

Following in a similar fashion as in [5] and using Lemmas 2.6, 2.7 and 2.10, we have $A = -1$ and $B = 1 + q \sum_{i=1}^{e-1} T^{-\frac{i(q-1)}{e}}(b)$.

Similarly, we calculate C and D ,

$$C = \frac{1}{(q-1)^3} \sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}T^l(b)T^n(a) \sum_{z \in \mathbb{F}_q^*} T^{l+m+n}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x)$$

and

$$D = \frac{1}{(q-1)^4} \sum_{l,m,n,k=0}^{q-2} G_{-l}G_{-m}G_{-n}G_{-k}T^l(b)T^n(a)T^k(-1) \sum_{z \in \mathbb{F}_q^*} T^{l+m+n+k}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x) \sum_{y \in \mathbb{F}_q^*} T^{ek}(y).$$

The innermost sum of D is non zero only if $k = \frac{i(q-1)}{e}, i = 0, 1, \dots (e-1)$. For $k = 0, D = -C$. Thus, we can write

$$D = -C + \frac{1}{(q-1)^3} \sum_{i=1}^{e-1} \sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}G_{-\frac{i(q-1)}{e}} T^l(b)T^n(a)T^{\frac{i(q-1)}{e}}(-1) \\ \times \sum_{z \in \mathbb{F}_q^*} T^{l+m+n+\frac{i(q-1)}{e}}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x).$$

Now we can write $D = -C + D'$, where

$$D' = \frac{1}{(q-1)^3} \sum_{i=1}^{e-1} \sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}G_{-\frac{i(q-1)}{e}} T^l(b)T^n(a)T^{\frac{i(q-1)}{e}}(-1) \sum_{z \in \mathbb{F}_q^*} T^{l+m+n+\frac{i(q-1)}{e}}(z) \sum_{x \in \mathbb{F}_q^*} T^{md+n}(x).$$

Here, the term $D' = 0$ unless $n = -md$ and $l = (d-1)(m + \frac{(e-i)(q-1)}{e(d-1)})$. Thus

$$D' = \frac{1}{(q-1)} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}}(-1) \sum_{m=0}^{q-2} G_{(d-1)(-m-\frac{(e-i)(q-1)}{e(d-1)})} G_{-m} G_{md} T^{(d-1)(m+\frac{(e-i)(q-1)}{e(d-1)})}(b) T^{-md}(a). \quad (3.2)$$

By putting the values of A, B, C and D in Eq. (3.1), we get

$$qN_{e,d} = q^2 + q \sum_{i=1}^{e-1} T^{-\frac{i(q-1)}{e}}(b) + D'. \quad (3.3)$$

If $d \geq 2$ is an even integer and $m \in \mathbb{Z}$, then Davenport-Hasse relation for $q \equiv 1 \pmod{d}, t \in \{1, -1\}$ gives

$$G_{dm} = \frac{G_m G_{m+\frac{q-1}{d}} G_{m+\frac{2(q-1)}{d}} \dots G_{m+\frac{(d-1)(q-1)}{d}}}{q^{\frac{d-2}{2}} G_{\frac{q-1}{2}} T^{\frac{(d-2)(q-1)}{8}}(-1) T^{-m}(d^d)}$$

and

$$G_{(d-1)(-m-\frac{(e-i)(q-1)}{e(d-1)})} = \frac{G_{-m-\frac{(e-i)(q-1)}{e(d-1)}} G_{-m-\frac{(2e-i)(q-1)}{e(d-1)}} \dots G_{-m-\frac{((d-1)e-i)(q-1)}{e(d-1)}}}{q^{\frac{d-2}{2}} T^{\frac{d(d-2)(q-1)}{8(d-1)}}(-1) T^{m+\frac{(e-i)(q-1)}{e(d-1)}}(d-1)^{d-1}}, \quad i = 1, 2, \dots, (e-1).$$

Now using G_{md} and $G_{(d-1)(-m-\frac{(e-i)(q-1)}{e(d-1)})}$ in Equation (3.2), we get

$$D' = \frac{1}{q^{d-2}(q-1)G_{\frac{q-1}{2}} T^{\frac{(d-2)(2d-1)(q-1)}{8(d-1)}}(-1)} \sum_{i=1}^{e-1} \frac{G_{-\frac{i(q-1)}{e}} T^{\frac{(e-i)(q-1)}{e}}(b) T^{\frac{i(q-1)}{e}}(-1)}{T^{\frac{(e-i)(q-1)}{e}}(d-1)} \\ \times \sum_{m=0}^{q-2} \left\{ G_m G_{-m} \left\{ G_{m+\frac{q-1}{d}} G_{-m-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{2(q-1)}{d}} G_{-m-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \dots \right. \\ \times \left\{ G_{m+\frac{(d-2)(q-1)}{2d}} G_{-m-\frac{(\frac{d-2}{2}e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{q-1}{2}} G_{-m-\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}} \right\} \\ \times \left. \left\{ G_{m+\frac{(d+2)(q-1)}{2d}} G_{-m-\frac{(\frac{d+2}{2}e-i)(q-1)}{e(d-1)}} \right\} \dots \left\{ G_{m+\frac{(d-1)(q-1)}{d}} G_{-m-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right\} T^m \left(\frac{b^{d-1} d^d}{(d-1)^{d-1} a^d} \right) \right\}.$$

By arranging these terms, we get

$$\begin{aligned}
 D' &= \frac{1}{q^{d-2}(q-1)G_{\frac{q-1}{2}}T^{\frac{(d-2)(2d-1)(q-1)}{8(d-1)}}(-1)} \sum_{i=1}^{e-1} \frac{G_{-i(q-1)}T^{\frac{(e-i)(q-1)}{e}}(b)T^{\frac{i(q-1)}{e}}(-1)}{T^{\frac{(e-i)(q-1)}{e}}(d-1)} \sum_{m=0}^{q-2} \left\{ G_{m+\frac{q-1}{2}}G_{-m} \right\} \\
 &\times \left\{ G_m G_{-m-\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{q-1}{d}} G_{-m-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{2(q-1)}{d}} G_{-m-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \cdots \\
 &\times \left\{ G_{m+\frac{(d-2)(q-1)}{2d}} G_{-m-\frac{(\frac{d-2}{2}e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{(d+2)(q-1)}{2d}} G_{-m-\frac{(\frac{d+2}{2}e-i)(q-1)}{e(d-1)}} \right\} \cdots \\
 &\times \left\{ G_{m+\frac{(d-1)(q-1)}{d}} G_{-m-\frac{(d-1)e-i)(q-1)}{e(d-1)}} \right\} T^m \left(\frac{b^{d-1}d^d}{(d-1)^{d-1}a^d} \right).
 \end{aligned}$$

By using Lemma 2.4 in the above equation, we have

$$\begin{aligned}
 D' &= \frac{q^2}{(q-1)G_{\frac{q-1}{2}}T^{\frac{(d-2)(2d-1)(q-1)}{8(d-1)}}(-1)} \sum_{i=1}^{e-1} G_{-i(q-1)}T^{\frac{i(q-1)}{e}} \left(-\frac{d-1}{b} \right) \sum_{m=0}^{q-2} G_{\frac{q-1}{2}}T^m(-1) \\
 &\times \begin{pmatrix} T^{m+\frac{q-1}{2}} \\ T^m \end{pmatrix} G_{-\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}} T^{m+\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}}(-1) \begin{pmatrix} T^m \\ T^{m+\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}} \end{pmatrix} G_{\frac{(id-e)(q-1)}{ed(d-1)}} \\
 &\times T^{m+\frac{(e-i)(q-1)}{e(d-1)}}(-1) \begin{pmatrix} T^{m+\frac{q-1}{d}} \\ T^{m+\frac{(e-i)(q-1)}{e(d-1)}} \end{pmatrix} G_{\frac{(id-2e)(q-1)}{ed(d-1)}} T^{m+\frac{(2e-i)(q-1)}{e(d-1)}}(-1) \begin{pmatrix} T^{m+\frac{2(q-1)}{d}} \\ T^{m+\frac{(2e-i)(q-1)}{e(d-1)}} \end{pmatrix} \cdots \\
 &\times G_{\frac{(id-\frac{d}{2}e-i)(q-1)}{ed(d-1)}} T^{m+\frac{(\frac{d-2}{2}e-i)(q-1)}{e(d-1)}}(-1) \begin{pmatrix} T^{m+\frac{(d-2)(q-1)}{2d}} \\ T^{m+\frac{(\frac{d-2}{2}e-i)(q-1)}{e(d-1)}} \end{pmatrix} G_{\frac{(id-\frac{d}{2}e-i)(q-1)}{ed(d-1)}} \\
 &\times T^{m+\frac{(\frac{d+2}{2}e-i)(q-1)}{e(d-1)}}(-1) \begin{pmatrix} T^{m+\frac{(d+2)(q-1)}{2d}} \\ T^{m+\frac{(\frac{d+2}{2}e-i)(q-1)}{e(d-1)}} \end{pmatrix} \cdots G_{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}} T^{m+\frac{((d-1)e-i)(q-1)}{e(d-1)}}(-1) \\
 &\times \begin{pmatrix} T^{m+\frac{(d-1)(q-1)}{d}} \\ T^{m+\frac{((d-1)e-i)(q-1)}{e(d-1)}} \end{pmatrix} T^m \left(\frac{b^{(d-1)}d^d}{(d-1)^{(d-1)}a^d} \right).
 \end{aligned}$$

By collecting the terms of G_s and T^s , we get

$$\begin{aligned}
 D' &= \frac{q^2}{(q-1)T^{\frac{(d-2)(2d-1)(q-1)}{8(d-1)}}(-1)} \sum_{i=1}^{e-1} M_i T^{\frac{i(q-1)}{e}} \left(-\frac{d-1}{b} \right) \sum_{m=0}^{q-2} T^{md+\frac{d(q-1)}{2}}(-1) \\
 &\times T^{-\frac{i(q-1)}{e}}(-1) \begin{pmatrix} T^{m+\frac{q-1}{2}} \\ T^m \end{pmatrix} \begin{pmatrix} T^m \\ T^{m+\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}} \end{pmatrix} \begin{pmatrix} T^{m+\frac{q-1}{d}} \\ T^{m+\frac{(e-i)(q-1)}{e(d-1)}} \end{pmatrix} \begin{pmatrix} T^{m+\frac{2(q-1)}{d}} \\ T^{m+\frac{(2e-i)(q-1)}{e(d-1)}} \end{pmatrix} \cdots \\
 &\times \begin{pmatrix} T^{m+\frac{(d-2)(q-1)}{2d}} \\ T^{m+\frac{\frac{d-2}{2}e-i)(q-1)}{e(d-1)}} \end{pmatrix} \begin{pmatrix} T^{m+\frac{(d+2)(q-1)}{2d}} \\ T^{m+\frac{(\frac{d+2}{2}e-i)(q-1)}{e(d-1)}} \end{pmatrix} \cdots \begin{pmatrix} T^{m+\frac{(d-1)(q-1)}{d}} \\ T^{m+\frac{((d-1)e-i)(q-1)}{e(d-1)}} \end{pmatrix} T^m \left(\frac{b^{d-1}d^d}{(d-1)^{d-1}a^d} \right),
 \end{aligned}$$

where

$$M_i = G_{-i(q-1)/e} G_{-\frac{(\frac{d}{2}e-i)(q-1)}{e(d-1)}} G_{\frac{(id-e)(q-1)}{ed(d-1)}} G_{\frac{(id-2e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-\frac{d}{2}(q-1)e)(q-1)}{ed(d-1)}} G_{\frac{id-(\frac{d}{2}+1)e(q-1)}{e(d-1)}} \cdots$$

$$\times G_{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}}, \quad i = 1, 2, \dots, (e-1).$$

Since d is even, thus $T^{md+\frac{d(q-1)}{2}}(-1) = 1$. By Definition 2.12, we have

$$D' = \frac{q}{T^{\frac{(d-2)(2d-1)(q-1)}{8(d-1)}}(-1)} \sum_{i=1}^{e-1} M_i T^{\frac{i(q-1)}{e}} \left(\frac{d-1}{b} \right)$$

$$\times {}_dF_{d-1} \left(\begin{matrix} \phi, & \epsilon, & \chi, & \chi^2, \dots, \chi^{\frac{d-2}{2}}, & \chi^{\frac{d+2}{2}}, \dots, \chi^{d-1} \\ \psi^{\frac{d}{2}e-i}, & \psi^{e-i}, & \psi^{2e-i}, \dots, \psi^{\frac{d-2}{2}e-i}, & \psi^{\frac{d+2}{2}e-i}, \dots, \psi^{(d-1)e-i} \end{matrix} \middle| \alpha \right),$$

where $\alpha = \frac{d}{a} \left(\frac{bd}{a(d-1)} \right)^{(d-1)}$, χ and ψ are characters of orders d and $e(d-1)$ respectively. We complete the proof by putting the value of D' in equation (3.3).

Proof of the Theorem 3.3. If $d \geq 3$ is an odd integer, then Davenport-Hasse relations for G_{dm} and $G_{(d-1)\left(-m-\frac{(e-1)(q-1)}{e(d-1)}\right)}$ are given by

$$G_{dm} = \frac{G_m G_{m+\frac{q-1}{d}} G_{m+\frac{2(q-1)}{d}} \cdots G_{m+\frac{(d-1)(q-1)}{d}}}{q^{\frac{d-1}{2}} T^{\frac{(d-1)(d+1)(q-1)}{8d}}(-1) T^{-m}(d^d)},$$

$$G_{(d-1)\left(-m-\frac{(e-1)(q-1)}{e(d-1)}\right)} = \frac{G_{-m-\frac{(e-i)(q-1)}{e(d-1)}} G_{-m-\frac{(2e-i)(q-1)}{e(d-1)}} \cdots G_{-m-\frac{((d-1)e-i)(q-1)}{e(d-1)}}}{G_{\frac{q-1}{2}} q^{\frac{d-3}{2}} T^{\frac{(d-3)(q-1)}{8}}(-1) T^{m+\frac{(e-i)(q-1)}{e(d-1)}}(d-1)^{(d-1)}}, \quad i = 1, 2, \dots, (e-1).$$

Using these identities in Eq. (3.2), we get

$$D' = \frac{T^{\frac{(3d-1)(q-1)}{8d}}(-1)}{(q-1)q^{(d-2)}G_{\frac{q-1}{2}}T^{\frac{(d^2-1)(q-1)}{4d}}(-1)} \sum_{i=1}^{e-1} G_{-i(q-1)/e} T^{\frac{i(q-1)}{e}}(-1) T^{\frac{(e-i)(q-1)}{e}} \left(\frac{b}{d-1} \right)$$

$$\times \sum_{m=0}^{q-2} \{G_m G_{-m}\} \left\{ G_{m+\frac{q-1}{d}} G_{-m-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{2(q-1)}{d}} G_{-m-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \cdots$$

$$\times \left\{ G_{m+\frac{(d-1)(q-1)}{2d}} G_{-m-\frac{(\frac{d-1}{2}e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{(d+1)(q-1)}{2d}} G_{-m-\frac{(\frac{d+1}{2}e-i)(q-1)}{e(d-1)}} \right\} \cdots$$

$$\times \left\{ G_{m+\frac{(d-1)(q-1)}{d}} G_{-m-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right\} T^m \left(\frac{b^{(d-1)}d^d}{(d-1)^{(d-1)}a^d} \right).$$

Since d is an odd integer, thus $T^{\frac{(d^2-1)(q-1)}{4d}}(-1) = 1$. Next, we eliminate the term $\{G_m G_{-m}\}$, using the fact that if $m = 0$, then $G_m G_{-m} = qT^m(-1) - (q-1)$, and if $m \neq 0$, then $G_m G_{-m} = qT^m(-1)$. Using these identities in the above equation and rearranging the second term, we have

$$\begin{aligned}
 D' &= \frac{qT^{\frac{(3d-1)(q-1)}{8d}}(-1)}{(q-1)q^{(d-2)}G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}} (-1) T^{\frac{(e-i)(q-1)}{e}} \left(\frac{b}{d-1}\right) \\
 &\times \sum_{m=0}^{q-2} \left\{ G_{m+\frac{q-1}{d}} G_{-m-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{2(q-1)}{d}} G_{-m-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \cdots \\
 &\times \left\{ G_{m+\frac{(d-1)(q-1)}{2d}} G_{-m-\frac{(\frac{d-1}{2}e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{m+\frac{(d+1)(q-1)}{2d}} G_{-m-\frac{(\frac{d+1}{2}e-i)(q-1)}{e(d-1)}} \right\} \cdots \\
 &\times \left\{ G_{m+\frac{(d-1)(q-1)}{d}} G_{-m-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right\} T^m \left(-\frac{b^{(d-1)}d^d}{(d-1)^{(d-1)}a^d} \right) \\
 &- \frac{T^{\frac{(3d-1)(q-1)}{8d}}(-1)}{q^{(d-2)}G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}} (-1) T^{\frac{(e-i)(q-1)}{e}} \left(\frac{b}{d-1}\right) \\
 &\times \left\{ G_{\frac{q-1}{d}} G_{-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{\frac{2(q-1)}{d}} G_{-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \cdots \left\{ G_{\frac{(d-1)(q-1)}{2d}} G_{-\frac{(\frac{d-1}{2}e-i)(q-1)}{e(d-1)}} \right\} \\
 &\times \left\{ G_{\frac{(d+1)(q-1)}{2d}} G_{-\frac{(\frac{d+1}{2}e-i)(q-1)}{e(d-1)}} \right\} \cdots \left\{ G_{\frac{(d-1)(q-1)}{d}} G_{-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right\}.
 \end{aligned}$$

By using Lemma 2.4 in each terms of the above equation and collecting the terms of G_s and T^s and $T^{m(d-1)}(-1) = 1$ (since d is an odd integer), we have

$$\begin{aligned}
 D' &= \frac{q^2 T^{\frac{(4d^2+3d-1)(q-1)}{8d}}(-1)}{(q-1)G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}} (-1) T^{-\frac{i(q-1)}{e}} (-1) T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1}\right) \\
 &\times \left\{ G_{\frac{(id-e)(q-1)}{ed(d-1)}} G_{\frac{(id-2e)(q-1)}{ed(d-1)}} G_{\frac{(id-3e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-\frac{(d-1)}{2}e)(q-1)}{ed(d-1)}} G_{\frac{(id-\frac{(d+1)}{2}e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}} \right\} \\
 &\times \sum_{m=0}^{q-2} \left(\frac{T^{m+\frac{q-1}{d}}}{T^{m+\frac{(e-i)(q-1)}{e(d-1)}}} \right) \left(\frac{T^{m+\frac{2(q-1)}{d}}}{T^{m+\frac{(2e-i)(q-1)}{e(d-1)}}} \right) \cdots \left(\frac{T^{m+\frac{(d-1)(q-1)}{2d}}}{T^{m+\frac{(\frac{d-1}{2}e-i)(q-1)}{e(d-1)}}} \right) \left(\frac{T^{m+\frac{(d+1)(q-1)}{2d}}}{T^{m+\frac{(\frac{d+1}{2}e-i)(q-1)}{e(d-1)}}} \right) \cdots \\
 &\times \left(\frac{T^{m+\frac{(d-1)(q-1)}{d}}}{T^{m+\frac{((d-1)e-i)(q-1)}{e(d-1)}}} \right) T^m(-a) - \frac{T^{\frac{(3d-1)(q-1)}{8d}}(-1)}{q^{(d-2)}G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}} (-1) T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1}\right) N_i,
 \end{aligned}$$

where

$$\begin{aligned}
 N_i &= \left\{ G_{\frac{q-1}{d}} G_{-\frac{(e-i)(q-1)}{e(d-1)}} \right\} \left\{ G_{\frac{2(q-1)}{d}} G_{-\frac{(2e-i)(q-1)}{e(d-1)}} \right\} \cdots \left\{ G_{\frac{(d-1)(q-1)}{2d}} G_{-\frac{(\frac{d-1}{2}e-i)(q-1)}{e(d-1)}} \right\} \\
 &\times \left\{ G_{\frac{(d+1)(q-1)}{2d}} G_{-\frac{(\frac{d+1}{2}e-i)(q-1)}{e(d-1)}} \right\} \cdots \left\{ G_{\frac{(d-1)(q-1)}{d}} G_{-\frac{((d-1)e-i)(q-1)}{e(d-1)}} \right\}, \quad i = 1, 2, \dots, (e-1),
 \end{aligned}$$

and $\alpha = \left(\frac{d}{a}\right) \left(\frac{bd}{a(d-1)}\right)^{(d-1)}$. Replace $m + \frac{(e-i)(q-1)}{e(d-1)}$ by m in the above equation, we have

$$\begin{aligned}
 D' &= q^2 \frac{T^{\frac{(4d^2+3d-1)(q-1)}{8d}} (-1)}{(q-1)G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1}\right) M_i \sum_{m=0}^{q-2} \binom{T^{m+\frac{(id-e)(q-1)}{ed(d-1)}}}{T^m} \\
 &\times \binom{T^{m+\frac{(id+(d-2)e)(q-1)}{ed(d-1)}}}{T^{m+\frac{e(q-1)}{e(d-1)}}} \cdots \binom{T^{m+\frac{(ed^2-4ed+2id+e)(q-1)}{2ed(d-1)}}}{T^{m+\frac{(d-3)e(q-1)}{2e(d-1)}}} \binom{T^{m+\frac{(ed^2-2ed+2id-e)(q-1)}{2ed(d-1)}}}{T^{m+\frac{(d-1)e(q-1)}{2e(d-1)}}} \cdots \\
 &\times \binom{T^{m+\frac{(ed^2-3ed+id+e)(q-1)}{ed(d-1)}}}{T^{m+\frac{(d-2)e(q-1)}{e(d-1)}}} T^m (-\alpha) T^{-\frac{(e-i)(q-1)}{e(d-1)}} (-\alpha) - \frac{T^{\frac{(3d-1)(q-1)}{8d}} (-1)}{q^{(d-2)} G_{\frac{q-1}{2}}} \\
 &\times \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}} (-1) T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1}\right) N_i,
 \end{aligned}$$

where

$$\begin{aligned}
 M_i &= \left\{ G_{\frac{(id-e)(q-1)}{ed(d-1)}} G_{\frac{(id-2e)(q-1)}{ed(d-1)}} G_{\frac{(id-3e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-\frac{d-1}{2}e)(q-1)}{ed(d-1)}} G_{\frac{(id-\frac{d+1}{2}e)(q-1)}{ed(d-1)}} \right. \\
 &\quad \left. \times G_{\frac{(id-\frac{d+3}{2}e)(q-1)}{ed(d-1)}} \cdots G_{\frac{(id-(d-1)e)(q-1)}{ed(d-1)}} \right\}, \quad i = 1, 2, \dots, (e-1).
 \end{aligned}$$

By using Definition 2.12 in the above equation, we have

$$\begin{aligned}
 D' &= -\frac{T^{\frac{(3d-1)(q-1)}{8d}} (-1)}{q^{(d-2)} G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} N_i G_{-\frac{i(q-1)}{e}} T^{\frac{i(q-1)}{e}} (-1) T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1}\right) \\
 &+ q \frac{T^{\frac{(4d^2+3d-1)(q-1)}{8d}} (-1)}{G_{\frac{q-1}{2}}} \sum_{i=1}^{e-1} G_{-\frac{i(q-1)}{e}} T^{-\frac{i(q-1)}{e}} \left(\frac{b}{d-1}\right) M_i T^{-\frac{(e-i)(q-1)}{e(d-1)}} (-\alpha) {}_{d-1}F_{d-2} \\
 &\times \left(\begin{array}{c} \eta^{id-e}, \eta^{id+ed-2e}, \dots, \eta^{\frac{ed^2-4ed+2id+e}{2}}, \eta^{\frac{ed^2-2ed+2id-e}{2}}, \dots, \eta^{ed^2-3ed+id+e} \\ \psi^e, \dots, \psi^{\frac{d-3}{2}e}, \qquad \qquad \qquad \psi^{\frac{d-1}{2}e}, \dots, \psi^{(d-2)e} \end{array} \middle| -\alpha \right),
 \end{aligned}$$

where η, ψ are characters of orders $ed(d-1)$ and $e(d-1)$ respectively. We complete the proof by putting the value of D' in equation (3.3). Note that if $e = 2$, then N_1 becomes $q^{d-1} T^{-\frac{(d-1)(q-1)}{8d}} (-1)$ and $M_1 = q^{\frac{d-1}{2}} T^{-\frac{(d-1)(q-1)}{8d}} (-1)$.

Remark 3.5. Let $\pi \in \mathbb{F}_p^*$ be of the order e . If $e = d$ and $p \equiv 1 \pmod{e}$, then there are e points at infinity, namely $[1 : 1 : 0], [1 : \pi : 0], [1 : \pi^2 : 0], \dots, [1 : \pi^{e-1} : 0]$. Again if $e = d$ and $p \not\equiv 1 \pmod{e}$, then the point at infinity is only $[1 : 1 : 0]$. Now if $e \neq d$ for $e < d$, the point at infinity is only $[0 : 1 : 0]$ and for $e > d$, then the point at infinity is only $[1 : 0 : 0]$.

Remark 3.6. Let q, e, d and $E_{e,d}$ be as in Theorems 3.1 and 3.3. Let $a_q(E_{e,d}(\mathbb{F}_q))$ denotes the trace of Frobenius of the algebraic curve $E_{e,d}$. Since $a_q(E_{e,d}(\mathbb{F}_q)) = q - N_{e,d}$, from Theorems 3.1 and 3.3, we can express the trace of Frobenius of the algebraic curve $E_{e,d}$ in terms of ${}_dF_{d-1}$ and ${}_{d-1}F_{d-2}$ Gaussian hypergeometric series containing multiplicative characters of orders $d, e(d-1)$ and $ed(d-1)$.

4. Applications

In the following example, we deduce Theorem 2.1 of Lennon [12] from Theorem 3.3.

Example 4.1 ([12]). Let $q = p^n$, $p > 3$ a prime and $q \equiv 1 \pmod{12}$, Let $E_{2,3} : y^2 = x^3 + ax + b$ be an elliptic curve over \mathbb{F}_q with $j(E_{2,3}) \neq 0, 1728$, then the trace of Frobenius map on $E_{2,3}$ can be expressed as

$$a_q(E_{2,3}) = -qT^{\frac{q-1}{4}} \left(\frac{a^3}{27} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ T^{\frac{q-1}{2}} \end{matrix} \middle| -\frac{27b^2}{4a^3} \right).$$

Let $e = 2$ and $d = 3$ in Theorem 3.3. In this case, we have $M_1 = qT^{-\frac{q-1}{12}}(-1)$, $N_1 = q^2T^{-\frac{q-1}{12}}(-1)$ and

$$\begin{aligned} N_{2,3} &= q + \phi(b) - T^{-\frac{q-1}{12}}(-1)\phi(-2b)T^{\frac{q-1}{3}}(-1) \\ &+ qT^{\frac{11(q-1)}{6}}(-1)\phi(2b)T^{-\frac{q-1}{12}}(-1)T^{\frac{q-1}{4}} \left(-\frac{27b^2}{4a^3} \right) {}_2F_1 \left(\begin{matrix} \eta, \eta^5 \\ \psi^2 \end{matrix} \middle| -\frac{27b^2}{4a^3} \right), \end{aligned}$$

where η is a multiplicative character of order 12 and ψ is multiplicative character of order 4. Thus, we have

$$N_{2,3} = q + \phi(b) - \phi(-2b)T^{\frac{q-1}{4}}(-1) + qT^{\frac{q-1}{4}} \left(\frac{a^3}{27} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27b^2}{4a^3} \right).$$

Since $q \equiv 1 \pmod{12}$, therefore, $\phi(2) = T^{\frac{q-1}{4}}(-1)$ and $\phi(-1) = 1$. Thus, $\phi(2b)T^{\frac{q-1}{4}}(-1) = \phi(b)$. Hence, we have

$$N_{2,3} = q + qT^{\frac{q-1}{4}} \left(\frac{a^3}{27} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27b^2}{4a^3} \right).$$

Since $a_q(E_{2,3}) = q - N_{2,3}$, we have

$$a_q(E_{2,3}) = -qT^{\frac{q-1}{4}} \left(\frac{a^3}{27} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27b^2}{4a^3} \right).$$

We now give an example to show how Theorem 3.3 is applied for specific values of e and d .

Example 4.2. If $q \equiv 1 \pmod{36}$ and $E_{3,4}$ is an algebraic curve over \mathbb{F}_q , then the trace of Frobenius map on $E_{3,4}$ can be expressed as

$$\begin{aligned}
 a_q(E_{3,4}(\mathbb{F}_q)) &= -T^{-\frac{q-1}{3}}(b) - T^{-\frac{2(q-1)}{3}}(b) - q^3 \left(\begin{matrix} T^{\frac{4(q-1)}{9}} \\ T^{\frac{q-1}{3}} \end{matrix} \right) \left(\begin{matrix} T^{\frac{q-1}{36}} \\ T^{\frac{5(q-1)}{36}} \end{matrix} \right) T^{\frac{q-1}{3}} \left(\frac{3}{b} \right) \\
 &\quad \times {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & T^{\frac{q-1}{4}}, & T^{\frac{3(q-1)}{4}} \\ & T^{\frac{5(q-1)}{9}}, & T^{\frac{2(q-1)}{9}}, & T^{\frac{8(q-1)}{9}} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right) - q^3 \left(\begin{matrix} T^{\frac{5(q-1)}{9}} \\ T^{\frac{2(q-1)}{3}} \end{matrix} \right) \left(\begin{matrix} T^{\frac{5(q-1)}{36}} \\ T^{\frac{q-1}{36}} \end{matrix} \right) \\
 &\quad \times T^{\frac{2(q-1)}{9}}(-1)T^{\frac{2(q-1)}{3}} \left(\frac{3}{b} \right) {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & T^{\frac{q-1}{4}}, & T^{\frac{3(q-1)}{4}} \\ & T^{\frac{4(q-1)}{9}}, & T^{\frac{q-1}{9}}, & T^{\frac{7(q-1)}{9}} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right).
 \end{aligned}$$

Let $e = 3$ and $d = 4$ in Theorem 3.1. In this case, we have

$$N_{3,4} = q + \sum_{i=1}^2 T^{-\frac{i(q-1)}{3}}(b) + T^{-\frac{7(q-1)}{12}}(-1) \sum_{i=1}^2 M_i T^{\frac{i(q-1)}{3}} \left(\frac{3}{b} \right) {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & \chi, & \chi^3 \\ & \psi^{6-i}, & \psi^{3-i}, & \psi^{9-i} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right), \tag{4.1}$$

where χ and ψ are characters of orders 4 and 9 respectively, and

$$M_i = \left\{ G_{-\frac{i(q-1)}{3}} G_{-\frac{(6-i)(q-1)}{9}} G_{\frac{(4i-3)(q-1)}{36}} G_{\frac{(4i-9)(q-1)}{36}} \right\}, \quad i = 1, 2.$$

Using Lemmas 2.3 and 2.4, we find the values of M_i for $i = 1$,

$$M_1 = q^3 \left(\begin{matrix} T^{\frac{4(q-1)}{9}} \\ T^{\frac{q-1}{3}} \end{matrix} \right) \left(\begin{matrix} T^{\frac{q-1}{36}} \\ T^{\frac{5(q-1)}{36}} \end{matrix} \right) T^{\frac{7(q-1)}{12}}(-1),$$

and for $i = 2$,

$$M_2 = q^3 \left(\begin{matrix} T^{\frac{5(q-1)}{9}} \\ T^{\frac{2(q-1)}{3}} \end{matrix} \right) \left(\begin{matrix} T^{\frac{5(q-1)}{36}} \\ T^{\frac{q-1}{36}} \end{matrix} \right) T^{\frac{29(q-1)}{36}}(-1).$$

By putting the value of M_i in equation (3.1), we have

$$\begin{aligned}
 N_{3,4} &= q + T^{-\frac{q-1}{3}}(b) + T^{-\frac{2(q-1)}{3}}(b) + q^3 \left(\begin{matrix} T^{\frac{4(q-1)}{9}} \\ T^{\frac{q-1}{3}} \end{matrix} \right) \left(\begin{matrix} T^{\frac{q-1}{36}} \\ T^{\frac{5(q-1)}{36}} \end{matrix} \right) T^{\frac{q-1}{3}} \left(\frac{3}{b} \right) \\
 &\quad \times {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & T^{\frac{q-1}{4}}, & T^{\frac{3(q-1)}{4}} \\ & T^{\frac{5(q-1)}{9}}, & T^{\frac{2(q-1)}{9}}, & T^{\frac{8(q-1)}{9}} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right) + q^3 \left(\begin{matrix} T^{\frac{5(q-1)}{9}} \\ T^{\frac{2(q-1)}{3}} \end{matrix} \right) \left(\begin{matrix} T^{\frac{5(q-1)}{36}} \\ T^{\frac{q-1}{36}} \end{matrix} \right) \\
 &\quad \times T^{\frac{2(q-1)}{9}}(-1)T^{\frac{2(q-1)}{3}} \left(\frac{3}{b} \right) {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & T^{\frac{q-1}{4}}, & T^{\frac{3(q-1)}{4}} \\ & T^{\frac{4(q-1)}{9}}, & T^{\frac{q-1}{9}}, & T^{\frac{7(q-1)}{9}} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right).
 \end{aligned}$$

Since $a_q(E_{3,4}) = q - N_{3,4}$, thus

$$\begin{aligned}
 a_q(E_{3,4}(\mathbb{F}_q)) &= -T^{-\frac{q-1}{3}}(b) - T^{-\frac{2(q-1)}{3}}(b) - q^3 \begin{pmatrix} T^{\frac{4(q-1)}{9}} \\ T^{\frac{q-1}{3}} \end{pmatrix} \begin{pmatrix} T^{\frac{q-1}{36}} \\ T^{\frac{5(q-1)}{36}} \end{pmatrix} T^{\frac{q-1}{3}} \left(\frac{3}{b}\right) \\
 &\quad \times {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & T^{\frac{q-1}{4}}, & T^{\frac{3(q-1)}{4}} \\ & & T^{\frac{5(q-1)}{9}}, & T^{\frac{2(q-1)}{9}}, & T^{\frac{8(q-1)}{9}} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right) - q^3 \begin{pmatrix} T^{\frac{5(q-1)}{9}} \\ T^{\frac{2(q-1)}{3}} \end{pmatrix} \begin{pmatrix} T^{\frac{5(q-1)}{36}} \\ T^{\frac{q-1}{36}} \end{pmatrix} \\
 &\quad \times T^{\frac{2(q-1)}{9}}(-1)T^{\frac{2(q-1)}{3}} \left(\frac{3}{b}\right) {}_4F_3 \left(\begin{matrix} \phi, & \epsilon, & T^{\frac{q-1}{4}}, & T^{\frac{3(q-1)}{4}} \\ & & T^{\frac{4(q-1)}{9}}, & T^{\frac{q-1}{9}}, & T^{\frac{7(q-1)}{9}} \end{matrix} \middle| -\frac{256b^3}{27a^4} \right).
 \end{aligned}$$

We recall the following theorems from [3,7].

Theorem 4.3 ([3]). Let $q = p^n$, $p > 0$ an odd prime and let T be a generator of the character group $\widehat{\mathbb{F}_q^*}$. The number of points on the twisted Edward curve $C_{\alpha,\beta} : ax^2 + y^2 = 1 + \beta x^2 y^2$ over \mathbb{F}_q can be expressed as

$$\#C_{\alpha,\beta}(\mathbb{F}_q) = q - 1 - \phi(\beta) - \phi(\alpha\beta) + q\phi(-\alpha) {}_2F_1 \left(\begin{matrix} \phi, & \phi \\ & \epsilon \end{matrix} \middle| \frac{\beta}{\alpha} \right).$$

Theorem 4.4 ([7]). If A, B and C are characters of \mathbb{F}_q , then

$${}_2F_1 \left(\begin{matrix} A, & \bar{A} \\ & \bar{AB} \end{matrix} \middle| \frac{1}{2} \right) = A(-2) \begin{cases} 0 & \text{if } B \text{ is not square} \\ \binom{C}{A} + \binom{\phi C}{A} & \text{if } B = C^2. \end{cases}$$

We have the following special case of the above theorem.

Corollary 4.5. If $A = \phi$, $B = \phi$ and $q \equiv 1 \pmod{4}$, then we have

$${}_2F_1 \left(\begin{matrix} \phi, & \phi \\ & \epsilon \end{matrix} \middle| \frac{1}{2} \right) = \phi(-2) \left[\binom{T^{\frac{q-1}{4}}}{\phi} + \binom{T^{\frac{3(q-1)}{4}}}{\phi} \right].$$

Theorem 4.6. Let $q = p^n$, $p > 0$ an odd prime and $q \equiv 1 \pmod{12}$. If $a, b, k \in \mathbb{F}_q^*$ and $3k + a = 0$, then the number of points on the elliptic curve $E_{a,b,0} : y^2 = x^3 + ax^2 + bx$ can be expressed as

$$\#E_{a,b,0} = q + qT^{\frac{3(q-1)}{4}} \left(\frac{3k^2 + 2ak + b}{3} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, & T^{\frac{5(q-1)}{12}} \\ & \phi \end{matrix} \middle| -\frac{27(k^3 + ak^2 + bk)^2}{4(3k^2 + 2ak + b)^3} \right).$$

Proof. Since $a \neq 0$, we find $k \in \mathbb{F}_q^*$ such that $3k + a = 0$. A change of variables $(x, y) \mapsto (x + k, y)$ takes the algebraic curve $E_{a,b,0}$ to birationally equivalent form $E'_{2,3} : y^2 = x^3 + (3k^2 + 2ak + b)x + (k^3 + ak^2 + bk)$. Clearly $\#E_{a,b,0} = \#E'_{2,3}$, using Example 4.1 for $a' = 3k^2 + 2ak + b$ and $b' = k^3 + ak^2 + bk$, we have

$$\#E_{a,b,0} = q + qT^{\frac{3(q-1)}{4}} \left(\frac{3k^2 + 2ak + b}{3} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27(k^3 + ak^2 + bk)^2}{4(3k^2 + 2ak + b)^3} \right).$$

□

Corollary 4.7. *Let $q = p^n$, $p > 0$ an odd prime and $q \equiv 1 \pmod{12}$. If $a, b, k \in \mathbb{F}_q^*$, b is a square, $3k + a = 0$ and $a \neq \pm 2\sqrt{b}$, then*

$$\begin{aligned} & qT^{\frac{3(q-1)}{4}} \left(\frac{3k^2 + 2ak + b}{3} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27(k^3 + ak^2 + bk)^2}{4(3k^2 + 2ak + b)^3} \right) \\ &= -\phi(a - 2\sqrt{b}) + \phi(ab - 2b\sqrt{b}) + q\phi(-a - 2\sqrt{b}) {}_2F_1 \left(\begin{matrix} \phi, \phi \\ \epsilon \end{matrix} \middle| \frac{a - 2\sqrt{b}}{a + 2\sqrt{b}} \right). \end{aligned}$$

Proof. A change of variables $(x, y) \mapsto \left(\frac{x}{y}, \frac{x - \sqrt{b}}{x + \sqrt{b}} \right)$ takes the algebraic curve $E_{a,b,0} : y^2 = x^3 + ax^2 + bx$ to birationally equivalent form $C_{\alpha,\beta} : \alpha x^2 + y^2 = 1 + \beta x^2 y^2$, where $\alpha = a + 2\sqrt{b}$ and $\beta = a - 2\sqrt{b}$. Now the points on $E_{a,b,0}$ for $y = 0$ and $x = -\sqrt{b}$ do not correspond to any points on $C_{\alpha,\beta}$. For $y = 0$, there are $2 + T^{\frac{q-1}{2}}(a^2 - 4b)$ extra points on $E_{a,b,0}$, and for $x = -\sqrt{b}$, there are $1 + T^{\frac{q-1}{2}}(ab - 2b\sqrt{b})$ extra points on $E_{a,b,0}$. Similarly under the inverse transformation, a change of variables $(x, y) \mapsto \left(\frac{\sqrt{b}(1+y)}{1-y}, \frac{\sqrt{b}(1+y)}{x(1-y)} \right)$ takes the algebraic curve $C_{\alpha,\beta}$ to birationally equivalent form $E_{a,b,0}$. Now the points on $C_{\alpha,\beta}$ for $x = 0$ and $y = 1$ do not correspond to any points on $E_{a,b,0}$, so for $x = 0$ and $y = 1$, there are two extra point $(0, 1), (0, -1)$ on $C_{\alpha,\beta}$. Clearly $\#E_{a,b,0} + 2 = \#C_{\alpha,\beta} + 3 + T^{\frac{q-1}{2}}(a^2 - 4b) + T^{\frac{q-1}{2}}(ab - 2b\sqrt{b})$. From Theorems 4.3 and 4.6, we have

$$\begin{aligned} & q + qT^{\frac{3(q-1)}{4}} \left(\frac{3k^2 + 2ak + b}{3} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27(k^3 + ak^2 + bk)^2}{4(3k^2 + 2ak + b)^3} \right) + 2 \\ &= q - 1 - \phi(a - 2\sqrt{b}) - \phi(a^2 - 4b) + 3 + \phi(a^2 - 4b) + \phi(ab - 2b\sqrt{b}) \\ &+ q\phi(-a - 2\sqrt{b}) {}_2F_1 \left(\begin{matrix} \phi, \phi \\ \epsilon \end{matrix} \middle| \frac{a - 2\sqrt{b}}{a + 2\sqrt{b}} \right). \end{aligned}$$

By solving the above equation, we have

$$\begin{aligned}
 & qT^{\frac{3(q-1)}{4}} \left(\frac{3k^2 + 2ak + b}{3} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27(k^3+ak^2+bk)^2}{4(3k^2+2ak+b)^3} \right) \\
 &= -\phi(a - 2\sqrt{b}) + \phi(ab - 2b\sqrt{b}) + q\phi(-a - 2\sqrt{b}) {}_2F_1 \left(\begin{matrix} \phi, \phi \\ \epsilon \end{matrix} \middle| \frac{a-2\sqrt{b}}{a+2\sqrt{b}} \right).
 \end{aligned}$$

□

In the next theorem, we obtain some special values of ${}_2F_1$ hypergeometric function containing characters of order 12.

Theorem 4.8. *Let $q = p^n$, p be a prime with $q \equiv 1 \pmod{12}$ and let T be a fixed generator of \mathbb{F}_q^* , then*

- (1) ${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| \frac{1323}{1331} \right) = T^{\frac{q-1}{4}} \left(\frac{-44}{3} \right) \phi(2) \left[\left(\begin{matrix} T^{\frac{q-1}{4}} \\ \phi \end{matrix} \right) + \left(\begin{matrix} T^{\frac{3(q-1)}{4}} \\ \phi \end{matrix} \right) \right],$
- (2) ${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \epsilon \end{matrix} \middle| \frac{8}{1331} \right) = T^{\frac{q-1}{3}} (-1) T^{\frac{q-1}{4}} \left(\frac{44}{3} \right) \phi(2) \left[\left(\begin{matrix} T^{\frac{q-1}{4}} \\ \phi \end{matrix} \right) + \left(\begin{matrix} T^{\frac{3(q-1)}{4}} \\ \phi \end{matrix} \right) \right],$
- (3) ${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{q-1}{12}} \\ \phi \end{matrix} \middle| \frac{-1323}{8} \right) = T^{\frac{q-1}{12}} \left(\frac{8}{1331} \right) T^{\frac{q-1}{4}} \left(\frac{-44}{3} \right) \phi(2) \left[\left(\begin{matrix} T^{\frac{q-1}{4}} \\ \phi \end{matrix} \right) + \left(\begin{matrix} T^{\frac{3(q-1)}{4}} \\ \phi \end{matrix} \right) \right],$
- (4) ${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{-5(q-1)}{12}} \\ \epsilon \end{matrix} \middle| \frac{-8}{1323} \right) = T^{\frac{q-1}{3}} (-1) T^{\frac{q-1}{12}} \left(\frac{1323}{1331} \right) T^{\frac{q-1}{4}} \left(\frac{44}{3} \right) \phi(2) \left[\left(\begin{matrix} T^{\frac{q-1}{4}} \\ \phi \end{matrix} \right) + \left(\begin{matrix} T^{\frac{3(q-1)}{4}} \\ \phi \end{matrix} \right) \right].$

Proof. Set $a = 12$ and $b = 4$ in Corollary 4.7, we have

$$\begin{aligned}
 & qT^{\frac{3(q-1)}{4}} \left(\frac{3k^2 + 24k + 4}{3} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{27(k^3+12k^2+4k)^2}{4(3k^2+24k+4)^3} \right) \\
 &= -\phi(8) + \phi(32) + q\phi(-16) {}_2F_1 \left(\begin{matrix} \phi, \phi \\ \epsilon \end{matrix} \middle| \frac{1}{2} \right).
 \end{aligned}$$

Since in Corollary 4.7, $3k + a = 0$, so k becomes -4 , then we have

$${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| \frac{1323}{1331} \right) = \phi(-1) T^{\frac{q-1}{4}} \left(\frac{-44}{3} \right) {}_2F_1 \left(\begin{matrix} \phi, \phi \\ \epsilon \end{matrix} \middle| \frac{1}{2} \right).$$

Using Corollary 4.5, the above equation becomes

$${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| \frac{1323}{1331} \right) = \phi(2) T^{\frac{q-1}{4}} \left(\frac{-44}{3} \right) \left[\left(T^{\frac{q-1}{4}} \right) + \left(T^{\frac{3(q-1)}{4}} \right) \right].$$

(2) Putting $x = \frac{1323}{1331} A = T^{\frac{q-1}{12}}$, $B = T^{\frac{5(q-1)}{12}}$ and $C = \phi$ in Theorem 2.13-(1), we obtain

$${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \epsilon \end{matrix} \middle| \frac{8}{1331} \right) = T^{\frac{q-1}{12}} (-1) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| \frac{1323}{1331} \right).$$

Thus the proof of (2) follows from the proof of (1).

(3) For $x = \frac{1323}{1331}$, $A = T^{\frac{q-1}{12}}$, $B = T^{\frac{5(q-1)}{12}}$ and $C = \phi$, Theorem 2.13-(2) yields

$${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| -\frac{1323}{8} \right) = T^{\frac{q-1}{12}} \left(\frac{8}{1331} \right) \phi(-1) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \phi \end{matrix} \middle| \frac{1323}{1331} \right).$$

Since $q \equiv 1 \pmod{12}$, therefore $\phi(-1) = 1$, then the proof of (3) follows from the proof of (1).

(4) Finally putting $x = \frac{8}{1331}$, $A = T^{\frac{q-1}{12}}$, $B = T^{\frac{5(q-1)}{12}}$ and $C = \epsilon$ in Theorem 2.13-(2), we obtain

$${}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \epsilon \end{matrix} \middle| -\frac{8}{1323} \right) = T^{\frac{q-1}{12}} \left(\frac{1323}{1331} \right) {}_2F_1 \left(\begin{matrix} T^{\frac{q-1}{12}}, T^{\frac{5(q-1)}{12}} \\ \epsilon \end{matrix} \middle| \frac{8}{1331} \right).$$

Hence the proof of (4) follows from the proof of (2). □

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