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Examples of non–autonomous basins of attraction–II

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Abstract. The aim of this article is to enlarge the list of examples of non–autonomous basins of attraction from [2] and at the same time explore some other properties that they satisfy. For instance, we show the existence of countably many disjoint *Short* $\mathbb{C}^k$'s in $\mathbb{C}^k$. We also construct a *Short* $\mathbb{C}^k$ which is not Runge and exhibit yet another example whose boundary has Hausdorff dimension $2k$.

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1. Introduction

We continue our work on non–autonomous basins of attraction, in particular on *Short* $\mathbb{C}^k$'s from [2]. Recall that a *Short* $\mathbb{C}^k$ is a proper subdomain $\Omega \subset \mathbb{C}^k$ satisfying the following properties:

(i) $\Omega = \bigcup \Omega_n$, where each Ω_n is biholomorphic to the ball $B^k(0; 1)$,
(ii) The infinitesimal Kobayashi metric vanishes identically, i.e., $k_\Omega \equiv 0$.
(iii) Ω admits a non–constant plurisubharmonic function that is bounded above.

These were first constructed by Fornæss [5] who showed that they can be obtained as non–autonomous basins of attraction of a sequence of automorphisms $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ of the form

$$F_n(z_1, z_2, \ldots, z_k) = (z_1^d + a_n z_k, a_n z_1, \ldots, a_n z_{k-1}) \tag{1.1}$$

where $0 < |a_{n+1}| < |a_n|^d < 1$ for every $n \geq 0$.

Recall that a sequence of automorphisms $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ is said to be *uniformly bounded* at the origin if there exists real constants $0 < C < D < 1$ and $r > 0$ such that

$$C\|z\| \leq \|F_n(z)\| \leq D\|z\|$$

for every $z \in B^k(0; r)$ and $n \geq 0$.

Note that the maps F_n in (1.1) are not uniformly bounded at the origin. The purpose of [2] was to give examples of *Short* $\mathbb{C}^k$'s that were motivated by the existing examples of Fatou–Bieberbach domains. Here we will extend this list by providing more examples of *Short* $\mathbb{C}^k$'s – in fact these examples will be biholomorphic images of non–autonomous basins of sequences of automorphisms satisfying the *uniform upper–bound* condition. Here, a sequence $\{F_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$ is said to satisfy the *uniform upper–bound* condition at the origin if there exists $r > 0$ and $0 < C < 1$ such that

$$\|F_n(z)\| < C\|z\|$$

for every $z \in B^k(0; r)$. Henceforth, the basin of attraction at the origin of such a sequence will always be denoted by $\Omega_{\{F_n\}}$. The results are organized in the following sequence:

In Section **2**, we show that there exist countably many disjoint *Short* $\mathbb{C}^k$'s in $\mathbb{C}^k$, $k \geq 2$ and there exist *Short* $\mathbb{C}^k$'s which are not Runge whenever $k \geq 4$. These results follow directly as an application of the fact that Fatou–Bieberbach domains with the aforementioned properties are known to exist from Rosay–Rudin [10] and Wold [12] respectively.

In Section **3**, we give two alternative methods to construct biholomorphic images of non–autonomous basins of attraction.

The first result here is specific to *Short* $\mathbb{C}^2$'s. However it can be proved for $k \geq 2$, from Remark 3.5. Now *Short* $\mathbb{C}^k$'s are constructed as the non–autonomous basin of attraction of sequence $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ where F_n's are as in (1.1). Our aim is to show that the sequence $\{F_n\}$ can involve higher order terms, i.e.,

$$F_n(z_1, z_2, \ldots, z_k) = (z_1^d + o(z_1^{d+1}) + a_n z_k, a_n z_1, \ldots, a_n z_{k-1})$$

where $0 < |a_{n+1}| < |a_n|^d < 1$ for every $n \geq 0$ and still the basin of attraction at the origin (i.e., $\Omega_{\{F_n\}}$) is a *Short* $\mathbb{C}^k$. Theorem **1.1** is achieved as an effort towards proving the same which is stated as follows:

Theorem 1.1. *Let $\{a_n\}$ be a strictly positive sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n^{-2^n} = 0$ and $0 < a_{n+1} < a_n^2 < 1$. If $\{F_n\} \subset \mathsf{Aut}_0(\mathbb{C}^2)$ of the form*

$$F_n(z_1, z_2) = (a_n z_2 + z_1^2 P(z_1), a_n z_1)$$

where P is a polynomial in one variable with positive coefficients and $P(0) = c_0 > 0$ then the basin of attraction at the origin (i.e., $\Omega_{\{F_n\}}$) is a Short $\mathbb{C}^2$.

However, it only says that z_1^d can be replaced with a polynomial in z_1 provided there is a restriction on the coefficients of the polynomial and the order of convergence of $|a_n|$'s.

Next, we show that a non–autonomous basins of attraction that satisfies the *uniform upper–bound* condition at a point is a parabolic domain realized as an increasing union of domains biholomorphic to the ball. Also, we give a sufficient condition for the existence of biholomorphisms between two such non–autonomous basin, i.e., given a sequence of automorphisms $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ satisfying the *uniform upper–bound* condition at the origin, each function can be sufficiently perturbed in a small enough ball at the origin so that the basin of attraction of the resulting sequence is biholomorphic to $\Omega_{\{F_n\}}$. This result is motivated from push–out methods due to Dixon–Esterle [3], Glovebnik [6] and Stensønes [11], [7].

Theorem 1.2. *Let $\{S_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$, $k \geq 2$ satisfy the uniform upper–bound condition and $\{F_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$. Then there exists a sequence of positive real numbers $\{\delta_n\}$, $\delta_n \to 0$ as $n \to \infty$ and $r > 0$ such that $\Omega_{\{F_n\}} \cong \Omega_{\{S_n\}}$ if*

$$\|F_n(z) - S_n(z)\| < \delta_n$$

for every $z \in B^k(0; r)$.

In Section **4**, Theorem **1.1** is applied to give a constructive proof of the existence of *Short* $\mathbb{C}^k$'s with chaotic boundary, i.e., there exists a *Short* $\mathbb{C}^k$ in $\mathbb{C}^k$ such that the upper box–dimension of the boundary is strictly greater than $2k - 1$ However, an existential proof of a much more stronger result will be achieved as an application of Theorem **1.2**.

In Sections **5** and **6**, we apply Theorem **1.2** to obtain results about biholomorphic images of non–autonomous basins of attraction at a point satisfying the *uniform upper–bound* condition. The analogs of these results for Fatou–Bieberbach domains are known to be true from [9], [13] and [12]. Our methods are adopted from the techniques in these articles. The new ingredient that we used is Theorem **1.2**. Here are our results:

Theorem 1.3. *Let K be a polynomially convex compact subset of $\mathbb{C}^k$ and let $\{p_j\} \subset \mathbb{C}^k \backslash K$ and $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ be a sequence that satisfies the uniform upper–bound condition at the origin. Then there exists a biholomorphism $\Phi : \Omega_{\{S_n\}} \to \mathbb{C}^k$, such that $\{p_j\} \subset \Phi(\Omega_{\{S_n\}}) \subset \mathbb{C}^k \backslash K$.*

Corollary 1.4. *Given a polynomially convex compact set K and a sequence of automorphisms $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ that satisfies the uniform upper–bound condition at the origin, there exists a biholomorphism $\Phi : \Omega_{\{S_n\}} \to \mathbb{C}^k$, such that $\Phi(\Omega_{\{S_n\}})$ is dense in $\mathbb{C}^k \backslash K$.*

Corollary 1.5. *Given a sequence of automorphisms $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ that satisfies the uniform upper–bound condition at the origin, there exists a biholomorphism $\Phi : \Omega_{\{S_n\}} \to \mathbb{C}^k$ such that $\Phi(\Omega_{\{S_n\}})$ is not Runge.*

Theorem 1.6. *Given any sequence of automorphisms $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ that satisfy the uniform upper–bound condition at the origin and any $m \in \mathbb{N} \cup \{\infty\}$ there exist m-biholomorphisms $\{\Phi_i : 1 \leq i \leq m\}$ such that the following hold:*

(i) $\Phi_i(\Omega_{\{S_n\}}) \cap \Phi_j(\Omega_{\{S_n\}}) = \emptyset$ *whenever* $1 \leq i \neq j \leq m$.
(ii) *Let* $\Omega = \cup\Phi_i(\Omega_{\{S_n\}})$. *For any* $q \in \mathbb{C}^k \backslash \Omega$, $q \in \partial\Phi_i(\Omega_{\{S_n\}})$ *for every* $1 \leq i \leq m$.

Theorem 1.7. *Given any sequence of automorphisms $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ that satisfy the uniform upper–bound condition at the origin, there exists a biholomorphism $\Phi : \Omega_{\{S_n\}} \to \mathbb{C}^k$ such that the Hausdorff dimension at any point in the boundary of $\Phi(\Omega_{\{S_n\}})$ is $2k$.*

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2. Examples of Short $\mathbb{C}^k$'s

First we prove that there exist countably many disjoint *Short* $\mathbb{C}^k$'s in $\mathbb{C}^k$. Recall that a sequence of points $\{p_j\}$ in $\mathbb{C}^k$ is said to be *tame*, if there exists an automorphism $\phi \in \mathsf{Aut}(\mathbb{C}^k)$ such that

$$\phi(p_j) = je_1$$

where $e_1 = (1, 0, \ldots, 0)$.

Proposition 2.1. *Given a tame sequence of point in $\mathbb{C}^k$, there exists a collection of disjoint Short $\mathbb{C}^k$'s centered at each point of the tame sequence.*

Proof. As noted in Rosay–Rudin [10], the automorphism F of $\mathbb{C}^2$ given by

$$F(z_1, z_2) = \left(z_1 + z_2, \frac{1}{2}(1 - z_2 - e^{z_1+z_2})\right)$$

has an attracting fixed point at each $p_m = (2m\pi i, 0)$ for every $m \geq 0$. Now, given a tame sequence, say $\{a_m\}$ in $\mathbb{C}^k$, $k \geq 2$, there exists an automorphism f_1 of $\mathbb{C}^k$ such that

$$f_1(a_m) = 2\pi i m e_1$$

where $e_1 = (1, 0, \ldots, 0)$. Let

$$f_2(z_1, z_2, \ldots, z_k) = (F(z_1, z_2), az_3, \ldots, az_k)$$

for $0 < |a| < 1$. This is an automorphism of $\mathbb{C}^k$. Clearly, $2\pi i m e_1$ is an attracting fixed point f_2 for each $m \geq 0$ and the corresponding attracting basin of f_2 for each $2\pi i m e_1$ (say Ω_m) is a Fatou–Bieberbach domain, i.e., there exist biholomorphisms $\psi_m : \Omega_m \to \mathbb{C}^k$ for every $m \geq 0$. Also, without loss of generality one can assume that $\psi_m(2\pi i m e_1) = 0$.

Now from [5], there exists a *Short* $\mathbb{C}^k$, say ω obtained as a non–autonomous of basin of attraction at the origin. Let $\omega_m = \psi_m^{-1}(\omega)$. Thus, ω_m is a *short* $\mathbb{C}^k$. Let $\Psi_m = f_1^{-1} \circ \psi_m^{-1}$ and $\widetilde{\omega}_m = \Psi_m(\omega)$. Then $\widetilde{\omega}_m$ is the required disjoint collection of *Short* $\mathbb{C}^k$'s. □

Let $\mathsf{Aut}_0(\mathbb{C}^k)$ denote the group of automorphisms of $\mathbb{C}^k$ that fixes the origin and for a sequence $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ let

$$F(n)(z) = F_n \circ \cdots \circ F_1(z).$$

Proposition 2.2. *Let $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$, $k \geq 2$ be a sequence of automorphisms such that the basin of attraction at $\Omega_{\{F_n\}}$ is a Short $\mathbb{C}^k$. Then for every $l \geq 1$, $\mathbb{C}^l \times \Omega_{\{F_n\}}$ is a Short $\mathbb{C}^{l+k}$.*

Proof. Since $\Omega_{\{F_n\}}$ is a *Short* $\mathbb{C}^k$ it satisfies the following properties:

(i) $\Omega_{\{F_n\}}$ is a non–empty open connected set of $\mathbb{C}^k$.
(ii) $\Omega_{\{F_n\}} = \cup_{j=1}^{\infty} \Omega_j$, $\Omega_j \subset \Omega_{j+1}$, and each Ω_j is biholomorphic to the unit ball $B^k(0;1)$ in $\mathbb{C}^k$. Further, for a given $0 < c < 1$ there exists $n_0 \geq 1$ such that

$$\Omega_j = F(n_0 + j)^{-1}(B^k(0;c))$$

(iii) The infinitesimal Kobayashi metric on $\Omega_{\{F_n\}}$ vanishes identically.
(iv) There exists a non–constant plurisubharmonic function $\phi : \Omega_{\{F_n\}} \to [-\infty, \infty)$ such that

$$\Omega_{\{F_n\}} = \{z \in \mathbb{C}^k : \phi(z) < 0\}.$$

Clearly $\mathbb{C}^l \times \Omega_{\{F_n\}}$ is an open connected set in $\mathbb{C}^{l+k}$. For each $n \geq 1$, let

$$\tilde{F}_n(z_1, z_2, \ldots, z_{k+l}) = (\alpha z_1, \alpha z_2, \ldots, \alpha z_l, F_n(z_{l+1}, \ldots, z_{k+l}))$$

where $0 < |\alpha| < 1$ and

$$\widetilde{\Omega}_j = \tilde{F}(n_0 + j)^{-1}\big(B^l(0;c) \times B^k(0;c)\big) = B^l(0; \alpha^{-(n_0+j)}c) \times \Omega_j.$$

Note that $\mathbb{C}^l \times \Omega_{\{F_n\}}$ is the basin of attraction of the sequence $\{\tilde{F}_n\}$ at the origin and

$$\mathbb{C}^l \times \Omega_{\{F_n\}} = \bigcup_{j \geq 0} \widetilde{\Omega}_j.$$

Let $U_j = \tilde{F}(j)^{-1}\big(B^{l+k}(0;c)\big)$. Then clearly, $U_{n_0+j} \subset \widetilde{\Omega}_j$. Since $\Omega_j \subset \Omega_{\{F_n\}}$, there exists $l_0 \geq 1$ such that for every $z \in \widetilde{\Omega}_j$

$$\tilde{F}(n_0 + j + l)(z) \in B^l(0; \alpha^{l_0}c) \times B^k(0; (c')^{l_0}c) \subset B^{l+k}(0;c)$$

where $0 < c' < 1$. Hence, $\widetilde{\Omega}_j \subset U_{n_0+j+l_0}$. Also for sufficiently large n, $U_n \subset U_{n+1}$ and thus $\mathbb{C}^l \times \Omega_{\{F_n\}} = \cup_{j \geq 0} U_{n_0+j+l_0}$.

Let $p \in \mathbb{C}^l \times \Omega_{\{F_n\}}$ and $\xi \in T_p(\mathbb{C}^l \times \Omega_{\{F_n\}})$. Let $p' = (p_1, \ldots, p_l)$, $p'' = ((p_{l+1}, \ldots, p_{l+k}), \xi' = (\xi_1, \ldots, \xi_l))$ and $\xi'' = (\xi_{l+1}, \ldots, \xi_{l+k})$. Since $\tilde{F}_n$ is a linear map in the first l-variables and $\Omega_{\{F_n\}}$ is a *Short* $\mathbb{C}^k$, there exist $F_1 : \Delta(0;1) \to \mathbb{C}^l$ such that

$$F_1(0) = p' \quad \text{and} \quad F_1'(0) = R\xi'$$

and $F_2 : \Delta(0;1) \to \Omega_{\{F_n\}}$ such that

$$F_2(0) = p'' \quad \text{and} \quad F_2'(0) = R\xi''$$

for every $R > 0$. Let $F(z) = (F_1(z), F_2(z))$. Thus $F(0) = p$ and $F'(0) = R\xi$. But this is true for any $R > 0$, and hence the infinitesimal Kobayashi metric vanishes on $\mathbb{C}^l \times \Omega_{\{F_n\}}$.

Let $\tilde{\phi}(z_1, \ldots, z_{k+l}) = \phi(z_{l+1}, \ldots, z_{k+l})$. Since $\tilde{\phi}$ is independent of the first l-variables, $\tilde{\phi}$ is plurisubharmonic on $\mathbb{C}^l \times \Omega_{\{F_n\}}$ and

$$\mathbb{C}^l \times \Omega_{\{F_n\}} = \{z \in \mathbb{C}^{l+k} : \tilde{\phi}(z) < 0\}.$$

Thus $\mathbb{C}^l \times \Omega_{\{F_n\}}$ is a non–autonomous of basin of attraction and is a *Short* $\mathbb{C}^{l+k}$. □

Corollary 2.3. *Let $\Omega_1 \subset \mathbb{C}^l$, $l \geq 2$ be a Fatou–Bieberbach domain and $\Omega_2 \subset \mathbb{C}^k$, $k \geq 2$ be a Short $\mathbb{C}^k$. Then $\Omega_1 \times \Omega_2$ is a Short $\mathbb{C}^{l+k}$.*

Proof. Let $\phi : \Omega_1 \to \mathbb{C}^l$ be a biholomorphism that identifies Ω_1 with $\mathbb{C}^l$. Then $\tilde{\phi} : \Omega_1 \times \Omega_2 \to \mathbb{C}^l \times \Omega_2$ as $\tilde{\phi}(z_1, z_2) = (\phi(z_1), z_2)$, for $z_1 \in \Omega_1$ and $z_2 \in \Omega_2$ is evidently a biholomorphism. Hence $\Omega_1 \times \Omega_2$ is a *Short* $\mathbb{C}^{l+k}$. □

Corollary 2.4. *For each $k \geq 4$ there exists a Short $\mathbb{C}^k$ which is not Runge.*

Proof. By Theorem **1** in [12], there exists a Fatou–Bieberbach domain (say Ω_1) in $\mathbb{C}^2$ which is not Runge. From Lemma **2.3**, if Ω_2 is a *Short* $\mathbb{C}^{k-2}$, then $\Omega_1 \times \Omega_2$ is *Short* $\mathbb{C}^k$.

Claim. $\Omega_1 \times \Omega_2$ is not Runge.

Since Ω_1 is not Runge there exists a compact set K such that the polynomial convex hull of K, $\widehat{K} \not\subset \Omega_1$. Now fix a $w_0 \in \Omega_2$ and define the following sets:

$$K_{w_0} = \{(z, w_0) \in \mathbb{C}^{l+k} : z \in K\} \quad \text{and} \quad \widehat{K}_{w_0} = \{(z, w_0) \in \mathbb{C}^{l+k} : z \in \hat{K}\}.$$

As $\widehat{K} \not\subset \Omega_1$, $\widehat{K}_{w_0} \not\subset \Omega_1 \times \Omega_2$. Suppose P be a polynomial map from $\mathbb{C}^{l+k}$ and $(z, w_0) \in \widehat{K}_{w_0}$. Then $P_{w_0}(z) = P(z, w_0)$ is polynomial in $\mathbb{C}^l$ and $|P_{w_0}(z)| \leq \|P_{w_0}\|_K$, i.e., $|P(z, w_0)| \leq \|P\|_{K_{w_0}}$. Hence $\widehat{K}_{w_0} \subset \widehat{K_{w_0}} \not\subset \mathbb{C}^{l+k}$. Thus the proof. □

Here is an alternative proof of the existence of a non-Runge *Short* $\mathbb{C}^k$, $k \geq 2$, that was suggested to us by Luka Boc–Thaler. Recall the following fact from [12].

Theorem 2.5. *There exists a subset $Y \subset \mathbb{C}^* \times \mathbb{C}$, such that $0 \in \widehat{Y}$. Further, for any $p \in \mathbb{C}^* \times \mathbb{C}$ and $\epsilon > 0$, there exists a biholomorphism of $\psi \in \mathsf{Aut}(\mathbb{C}^* \times \mathbb{C})$ such that $\psi(Y) = B^2(p; \epsilon)$.*

Proposition 2.6. *For every $k \geq 2$, there exists a Short $\mathbb{C}^k$ which is not Runge.*

Proof. From [12] there exists a Fatou–Bieberbach domain, D which is contained in $\mathbb{C}^* \times \mathbb{C}^{k-1}$. Let $\phi : D \to \mathbb{C}^k$, be the biholomorphism, then $\omega = \phi(\Omega)$ is a *Short* $\mathbb{C}^k$ where Ω is a *Short* $\mathbb{C}^k$. Now from Theorem **2.5**, there exist an automorphism, ψ of $\mathbb{C}^* \times \mathbb{C}^{k-1}$ such that for $B^k(p; \epsilon) \subset \omega$, $\psi(B^k(p; \epsilon)) = Y$. Hence $\psi(\omega)$ is a non–Runge *Short* $\mathbb{C}^k$. □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Note that there exists $M > 0$ such that

$$|z_1^2 P(z_1)| < M|z_1|^2 \quad \text{for every} \quad z_1 \in D(0; r)$$

whenever $0 < r < 1$. Let $0 < c < 1$ such that $(1 - Mc) > 0$. Further, choose $0 < c' < c$ and let $c_l = c(c')^l$.

Claim. For every $n \geq n_0$, sufficiently large and $l \geq 0$ then $F_{n+l}(\Delta^2(0; c_l)) \subset \Delta^2(0; c_{l+1})$.

Since $\lim_{n\to\infty} a_n^{-2^n} = 0$ there exists $0 < a < 1$ such that for n sufficiently large and every $l \geq 0$

$$\begin{aligned} \log a_{n+l} &= 2^{n+l} \log a_{n+l}^{-2^{n+l}} \\ &\leq (l + 1)2^n \log a. \end{aligned}$$

Thus, for some $n \geq n_0$ and every $l \geq 0$

$$\begin{aligned} \log a_{n+l} &< \log c(1 - Mc) + (l + 1) \log c' \\ a_{n+l} &< c(1 - Mc)c'^{l+1} < c(c')^{l+1} - M(c(c')^l)^2 < c_{l+1} - Mc_l^2 \\ \text{i.e.,} \quad a_{n+l}c_l + Mc_l^2 &< c_{l+1}. \end{aligned}$$

Hence the claim.

Now define

$$\Omega_n = \{z \in \mathbb{C}^2 : F(n)(z) \in \Delta^2(0; c)\}.$$

From the above claim, $\Omega_n \subset \Omega_{n+1}$ for sufficiently large n. Also for every $n \geq n_0$ and $l \geq 1$, $F_{n+l} \circ \cdots \circ F_{n+1}(z) \to 0$ uniformly on Ω_n thus $\cup_{n \geq n_0} \Omega_n \subset \Omega_{\{F_n\}}$. Conversely, if $z \in \Omega_{\{F_n\}}$, then $\|F(n)(z)\| < c$ for sufficiently large n, i.e., $z \in \Omega_n$ for n large. Hence $\cup_{n \geq n_0} \Omega_n = \Omega_{\{F_n\}}$. □

Lemma 3.1. *$\Omega_{\{F_n\}} = \cup_{j \geq 0} U_j$ where $U_j \subset U_{j+1}$ and each U_j is biholomorphic to the unit ball in $\mathbb{C}^2$. Further, the infinitesimal Kobayashi metric vanishes identically on $\Omega_{\{F_n\}}$.*

Proof. The proof is similar to the proof of (ii) and (iii) of Theorem 1.4 from [5]. □

Let $F(n)(z) = (f_1^n(z), f_2^n(z))$. Define

$$\phi_n(z) = \max\{|f_1^n(z)|, |f_2^n(z)|, a_n\}.$$

Lemma 3.2. *Let*

$$\psi_n = \frac{1}{2^n} \log \phi_n.$$

Then $\psi_n \to \psi$ on $\Omega_{\{F_n\}}$ and ψ is a plurisubharmonic function on $\Omega_{\{F_n\}}$.

Proof. Since $z \in \Omega_{\{F_n\}}$, there exists $n \geq n_z$ such that $\phi_n(z) \leq c$. Since $a_{n+1} \leq a_n^2$,

$$\phi_{n+1}(z) \leq \max\{M\phi_n(z)^2 + a_{n+1}, a_{n+1}\} \leq (M+1)\phi_n(z)^2.$$

Thus for every $z \in \Omega_{\{F_n\}}$

$$\frac{1}{2^{n+1}} \log \phi^{n+1}(z) \leq \frac{1}{2^{n+1}} \log M + \frac{1}{2^n} \log \phi_n(z).$$

Now define

$$\Phi_n(z) = \frac{1}{2^n} \log \phi_n(z) + \sum_{j \geq n} \frac{1}{2^{j+1}} \log M.$$

Thus Φ_n is a monotonically decreasing sequence of plurisubharmonic functions and hence its limit, i.e., ψ will be plurisubharmonic. □

Lemma 3.3. *For every $z \in \Omega_{\{F_n\}}$, $\psi(z) < 0$ i.e., ψ is a bounded plurisubharmonic function on $\Omega_{\{F_n\}}$. Further, $\Omega_{\{F_n\}}$ is not all of $\mathbb{C}^2$.*

Proof. The proof is similar to Lemma 3.3 and Lemma 3.4 in [2]. □

Lemma 3.4. *ψ is non–constant on $\Omega_{\{F_n\}}$.*

Proof. Since $\{F_n\} \subset \mathsf{Aut}_0(\mathbb{C}^2)$, $\psi(0) = -\infty$. If ψ is constant, then $\psi \equiv -\infty$ on $\Omega_{\{F_n\}}$.

Induction statement: For $x > 0$ and $y > 0$, $f_i^n(x, y) > 0$ for every $i = 1, 2$ and

$$\phi_n(x, y) > f_1^n(x, y) > c_0^{-1}(c_0 x)^{2^{n+1}}$$

for every $n \geq 0$.

Initial step: It is true since, $\pi_1 \circ F_0(x, y) = a_0 y + x^2 P(x) > c_0 x^2 > 0$ and $\pi_2 \circ F_0(x, y) = a_0 x > 0$.

General step: Assume that $f_1^n(x, y) > c_0^{-1}(c_0 x)^{2^{n+1}} > 0$ and $f_2^n(x, y) > 0$. Let

$$f_1^n(x, y) = c_0^{-1}(c_0 x)^{2^{n+1}} + c_n$$

for some $c_n > 0$. Now

$$f_1^{n+1}(x, y) = a_{n+1} f_2^n(x, y) + (c_0^{-1}(c_0 x)^{2^{n+1}} + c_n)^2 P(f_1^n(x, y)) > c_0^{-1}(c_0 x)^{2^{n+2}} > 0$$

and

$$f_2^{n+1}(x, y) = a_{n+1} f_1^n(x, y) > 0.$$

Since $\Delta^2(0; c) \subset \Omega_{\{F_n\}}$ for $x, y > 0$ and $(x, y) \in \Delta^2(0; c)$ it follows that

$$\psi_n(x, y) = \frac{\log c_0^{-1}}{2^n} + \log c_0 x \to \log c_0 x \neq -\infty$$

as $n \to \infty$. Hence ψ is non–constant on $\Omega_{\{F_n\}}$.

This completes the proof of Theorem 1.1. □

Remark 3.5. Let

$$S_n(z_1, z_2, \ldots, z_k) = (z_1^2 P(z_1) + a_n z_k, a_n z_1, \ldots, a_n z_{k-1})$$

be a sequence of shift–like maps in $\mathbb{C}^k$, $k \geq 3$, where P and $\{a_n\}$ are as in Theorem **1.1**. The same techniques can be adapted to prove that $\Omega_{\{S_n\}}$ (the basin of attraction of S_n's at the origin) is a *Short* $\mathbb{C}^k$.

Next, we prove a few properties of a non–autonomous basin of attraction, satisfying the *uniform upper–bound* condition.

Proposition 3.6. *Let $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$, $k \geq 2$ with uniform upper–bound condition at the origin. Then $\Omega_{\{S_n\}}$ satisfies the following properties:*

(i) *$\Omega_{\{S_n\}}$ is a connected open set in $\mathbb{C}^k$.*

(ii) *There exists $r_0 > 0$ such that for every $0 < r \leq r_0$, $\Omega_n^S \subset \Omega_{n+1}^S$ and*

$$\Omega_{\{S_n\}} = \bigcup_{n \geq 0} \Omega_n^S$$

where $\Omega_n^S = S(n)^{-1}(B^k(0; r))$.

(iii) *The infinitesimal Kobayashi metric vanishes identically on $\Omega_{\{S_n\}}$.*

Proof. By assumption there exist $r_0 > 0$ and $C < 1$ such that

$$\|S_n(z)\| \leq C\|z\|$$

for every $z \in \overline{B^k(0; r_0)}$ and $n \geq 0$. Further, for every $0 < r \leq r_0$, $B^k(0; r) \subset S_n^{-1}(B^k(0; r))$. Hence $\Omega_n^S \subset \Omega_{n+1}^S$. Similar arguments as in the proof of Theorem **1.1** gives $\Omega_{\{S_n\}} = \cup_{n \geq 0} \Omega_n^S$. This proves (i) and (ii).

Fix $p \in \Omega_{\{S_n\}}$ and $\xi \in T_p \Omega_{\{S_n\}}$, then for $p_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, where $p_n = F(n)(p)$ and $\xi_n = DF(n)\xi$. For any $R > 0$, consider the maps from unit disc, $\eta_n : \Delta(0; 1) \to \mathbb{C}^k$ defined as $\eta_n(x) = p_n + x R \xi_n$. Let $\tau_n = F(n)^{-1} \circ \eta_n$. Since $\eta_n(\Delta(0; 1)) \subset B^k(0; r)$ for n sufficiently large, $\tau_n(\Delta(0; 1)) \subset \Omega_{\{S_n\}}$. Now $\tau_n(0) = p$ and $\tau_n'(0) = R\xi$. As $R > 0$ is arbitrary, (iii) is true. □

Finally, we prove Theorem **1.2**.

Proof of Theorem 1.2. By assumption there exist $0 < r < 1$ and $C < 1$ such that

$$\|S_n(z)\| < C\|z\|$$

for every $z \in \overline{B^k(0;r)}$ and $n \geq 0$, i.e., $S_n(\overline{B^k(0;r)}) \subset B^k(0;Cr)$ for all $n \geq 0$. Let $r_0 = Cr$ and $0 < \epsilon < r - Cr < 1$. Then

$$B^k(0; r_0 + \epsilon) \subset B^k(0; r) \subset S_n^{-1}(B^k(0; r_0)) \tag{3.1}$$

for every $n \geq 0$. Let

$$M_n = \max\{\|D(S(n-1))^{-1}(z)\|_{\text{op}} : z \in B^k(0; r)\}$$

for every $n \geq 0$ and choose $0 < \epsilon_n < \epsilon^{n+1}/M_n$. Further, let $0 < \delta < \min\{\epsilon, 1 - C\}$ and $\tilde{C} = C + \delta < 1$. Thus

$$F_n(B^k(0; r)) \subset B^k(r_0 + \delta_n) \subset B^k(0; r). \tag{3.2}$$

By continuity of the functions S_n^{-1}'s there exists $\tilde{\delta}_n > 0$ such that

$$\|S_n^{-1}(z) - S_n^{-1}(w)\| < \epsilon_n$$

whenever $\|z - w\| < \tilde{\delta}_n$ for $z, w \in B^k(0; r)$. Let

$$\delta_n = \min\{\delta\tilde{C}^n r_0, \tilde{\delta}_n\}.$$

As $\|F_n(z) - S_n(z)\| < \delta_n$ for every $z \in \overline{B^k(0;r)}$ and $r_0 + \delta_n < r_0 + \epsilon < r < 1$ for every $n \geq 0$

$$\begin{aligned} &\|z - S_n^{-1} \circ F_n(z)\| < \epsilon_n \ \text{ for every } \ z \in B^k(0; r_0) \\ \text{i.e.,} \quad &\|F_n^{-1}(z) - S_n^{-1}(z)\| < \epsilon_n \ \text{ for every } \ z \in F_n^{-1}(B^k(0; r_0)). \end{aligned} \tag{3.3}$$

Claim. $\overline{B^k(0;r_0)} \subset\subset F_n^{-1}(B^k(0; r_0))$ for every $n \geq 0$.

From (3.1) and (3.3), it follows that $S_n^{-1}(B^k(0; r_0))$ is contained in an ϵ_n-neighbourhood of $F_n^{-1}(B^k(0; r_0))$, i.e.,

$$B^k(0; r) \subset\subset S_n^{-1}(B^k(0; r_0)) \subset (F_n^{-1} B^k(0; r_0)))_{\epsilon_n}.$$

But

$$B^k(0; r_0)_{\epsilon_n} = B^k(0; r_0 + \epsilon_n) \subset B^k(0; r)$$

Hence for every $n \geq 0$,

$$\overline{B^k(0; r_0)} \subset B^k(0; r - \epsilon_n) \subset\subset F_n^{-1}(B^k(0; r_0)).$$

Thus (3.3) is true for every $z \in \overline{B^k(0;r_0)}$. Further, by the choice of δ_n

$$\|S_n(z) - F_n(z)\| < \delta\tilde{C}^n r_0 \tag{3.4}$$

for every $z \in \overline{B^k(0;r_0)}$. For $z \in \partial B^k(0; r_0)$ then $\|S_n(z)\| < Cr_0$ hence

$$\|F_n(z)\| < Cr_0 + \delta r_0 < \tilde{C} r_0. \tag{3.5}$$

Induction hypothesis: If $z \in \overline{B^k(0;r_0)}$, then $F(n)(z) \in B^k(0; \tilde{C}^{n+1} r_0)$.

Initial step: From (3.5) note that $F_0(z) \in B^k(0; \tilde{C} r_0)$ for $z \in B^k(0; r_0)$.

General step: Suppose the claim is true for some $n \geq 0$. Let $z \in \partial B^k(0; \tilde{C}^{n+1} r_0)$. From (3.4)

$$\|F_{n+1}(z)\| < C\tilde{C}^{n+1} r_0 + \delta \tilde{C}^{n+1} r_0 \leq \tilde{C}^{n+2} r_0.$$

Hence $F_{n+1}(B^k(0; \tilde{C}^{n+1} r_0)) \subset B^k(0; \tilde{C}^{n+2} r_0)$, i.e., $F(n+1)(B^k(0; r_0)) \subset B^k(0; \underline{\tilde{C}^{n+2} r_0)}.$

Thus $B^k(0; r_0) \subset \Omega_{\{F_n\}}$. Also by similar arguments it follows that for every $z \in \overline{B^k(0; r_0)}$ and $0 \leq i \leq n$

$$F_{n+i} \circ F_{n+i-1} \circ \cdots \circ F_i(z) \in B^k(0; \tilde{C}^{n+1} r_0). \tag{3.6}$$

Let $\Omega_n^F = F(n)^{-1}(B^k(0; r_0))$. Now from the above claim, (3.6) and (3.5), $\Omega_n^F \subset \Omega_{n+1}^F$ and

$$\Omega_{\{F_n\}} = \bigcup_{n=0}^{\infty} \Omega_n^F.$$

So $\Omega_{\{F_n\}}$ is a connected open set containing the origin.

Let $\phi_n(z) = S(n)^{-1} F(n)(z) \in \mathsf{Aut}_0(\mathbb{C}^k)$.

Claim. $\phi_n \to \phi$ on compact subsets of $\Omega_{\{F_n\}}$.

Suppose K is a compact subset of $\Omega_{\{F_n\}}$, it is enough to show that for a given $\eta > 0$ there exists $n_0 \geq 0$ such that

$$\|\phi_n(z) - \phi_m(z)\| < \eta$$

for every $z \in K$ and $n, m \geq n_0$.

Choose $n_0 \geq \max\{n_1, n_2\}$ where $\epsilon^{n_1} < \eta(1 - \epsilon)$ and $K \subset \Omega_n^F$ for every $n \geq n_2$. Then

$$\|\phi_n(z) - \phi_m(z)\| \leq \sum_{i=n}^{m-1} \|\phi_{i+1}(z) - \phi_i(z)\| \leq \sum_{i=n}^{m-1} \|S(i+1)^{-1} F(i+1)(z) - S(i)^{-1} F(i)(z)\|$$

for every $z \in K$. Now $F(i)(z) \in B^k(0; r_0)$ for every $n \leq i \leq m-1$, i.e.,

$$\|S_{i+1}^{-1} \circ F(i+1)(z) - F(i)(z)\| = \|(S_{i+1}^{-1} \circ F_{i+1} - \mathrm{Id})(F(i)(z))\| < \epsilon_{i+1}.$$

Thus $S_{i+1}^{-1} \circ F(i+1)(z) \in B^k(0; r)$ for every $z \in K$ and

$$\|S(i+1)^{-1} F(i+1)(z) - S(i)^{-1} F(i)(z)\| \leq M_i \epsilon_{i+1} < \epsilon^{i+1}.$$

Hence, for every $z \in K$

$$\|\phi_n(z) - \phi_m(z)\| \leq \frac{\epsilon^{n+1}}{1 - \epsilon} < \eta.$$

Since ϕ_n converges uniformly on compact subset of $\Omega_{\{F_n\}}$, ϕ is holomorphic on $\Omega_{\{F_n\}}$.

Claim. ϕ is injective on $\Omega_{\{F_n\}}$.

Since ϕ is the limit of injective maps, an application of Hurwitz's Theorem shows that either, ϕ is injective or $\phi(\mathbb{C}^k)$ has empty interior. Let $\Omega_n^S = S(n)^{-1}(B^k(0; r_0))$ then from Proposition **3.6**, $\Omega_{\{S_n\}} = \cup_{n=0}^{\infty} \Omega_n^S$. Further, $\phi_n(\Omega_n^F) = \Omega_n^S$ for every $n \geq 0$. By uniform convergence of ϕ_n's on relatively compact subsets of $\Omega_{\{F_n\}}$, for a sufficiently small $0 < \eta < r_0$ there exists n sufficiently large

$$B^k(0; r_0) \subset \Omega_n^S \subset (\phi(\Omega_n^F))_\eta.$$

Here $(\phi(\Omega_n^F))_\eta$ is an η-neighbourhood of $\phi(\Omega_n^F)$. Now if interior of $\phi(\Omega_{\{F_n\}})$ is empty then from above condition $B^k(0; r_0) \subset B^k(0; \eta)$, which is a contradiction! Hence the claim.

Claim. $\phi(\Omega_{\{F_n\}}) = \Omega_{\{S_n\}}$.

Suppose $z = \phi(w)$ for some $w \in \Omega_{\{F_n\}}$. Let $z_n = \phi_n(w)$, i.e., $z_n \in \Omega_n^S$ for n sufficiently large. Now $z_n \to z$, i.e., $z \in \Omega_{\{S_n\}}$ or $z \in \partial\Omega_{\{S_n\}}$. Let $z \in \partial\Omega_{\{S_n\}}$. Since ϕ is injective there exists $z_0 \notin \overline{\Omega_{\{S_n\}}}$ such that $z_0 \in \phi(\Omega_{\{F_n\}})$ but arguments similar as above should give $z_0 \in \Omega_{\{S_n\}}$ or $z_0 \in \partial\Omega_{\{S_n\}}$. This is a contradiction! Thus $\phi(\Omega_{\{F_n\}}) \subset \Omega_{\{S_n\}}$.

Suppose $z \in \Omega_{\{S_n\}}$ and $z \notin \phi(\Omega_{\{F_n\}})$ then there exists $\rho > 0$ such that $\overline{B^k(z;\rho)} \cap \phi(\Omega_{\{F_n\}}) = \emptyset$ and $\overline{B^k(z;\rho)} \subset \Omega_{\{S_n\}}$, i.e.,

$$\overline{B^k(z;\rho)} \subset \Omega_n^S = \phi_n(\Omega_n^F)$$

for $n \geq n_0$. For n sufficiently large

$$\phi(\Omega_n^F) \subset (\Omega_n^S)_\eta \quad \text{and} \quad \Omega_n^S \subset (\phi(\Omega_n^F))_\eta$$

for $0 < \eta < \rho$. But by choice $z \notin \phi(\Omega_{\{F_n\}})_\eta$, i.e., $z \notin \Omega_n^S$ for every $n \geq 0$ which is a contradiction! Hence $\phi(\Omega_{\{F_n\}}) = \Omega_{\{S_n\}}$. □

Remark 3.7. The choice of δ_n in the proof of Theorem **1.2**, depends on the radius of the ball, i.e., $r > 0$ where $\{S_n\}$ satisfies the *uniform upper–bound* condition. However, the choice of $\delta_n(\tilde{r})$ can be appropriately modified whenever $0 < \tilde{r} < r$ to give that $\Omega_{\{F_n\}} \cong \Omega_{\{S_n\}}$ if

$$\|F_n(z) - S_n(z)\| < \delta_n(\tilde{r})$$

for every $z \in B^k(0;\tilde{r})$.

4. Short $\mathbb{C}^k$s with boundary having upper–box dimension greater than $2k-1$

Lemma 4.1. *Let P be a hyperbolic polynomial and $J_P(\delta_0)$ denote the δ_0-neighbourhood of the Julia set of J_P, then there exists $\{c_n'\}$ a sequence positive real numbers converging to* 0 *such that if $|w_n| \leq c_n'$ and $z_0 \in \mathbb{C}\backslash P^{-1}(J_P(\delta_0))$ then as $n \to \infty$, either*

$$P(z_n) + w_n \to 0 \quad \textit{or} \quad P(z_n) + w_n \to \infty$$

where $z_n = P(z_{n-1}) + w_{n-1}$ for $n \geq 1$.

Proof. Suppose z_0 lies in a compact component of $\mathbb{C}\backslash P^{-1}(J_P(\delta_0))$, say C. Let $\delta_1 > 0$ be chosen such that the $2\delta_1$ neighbourhood of $P(C)$, i.e., $P(C)_{2\delta_1} \subset C$. Let $C_1 = P(C)_{\delta_1}$, and similarly choose $\delta_2 > 0$ such that $P(C_1)_{2\delta_2} \subset C_1$. Now inductively define $C_n = P(C_{n-1})_{\delta_n}$ for $n \geq 2$ where $\delta_n > 0$ is appropriately chosen to satisfy

$$P(C_n)_{2\delta_{n+1}} \subset C_n.$$

Clearly $\operatorname{diam}(C_n) \to 0$. Hence for $z_0 \in C$ and $|w_n| < \delta_n$, the sequence $z_n \to 0$ as $n \to \infty$.

A similar argument on the non–compact component of $\mathbb{C}\backslash P^{-1}(J_P(\delta_0))$ gives a sequence η_n such that if $|w_n| < \eta_n$, then $z_n \to \infty$ as $n \to \infty$. Finally choose $c_n' < \min\{\delta_n, \eta_n\}$ for every $n \geq 1$. □

Let p be as in Theorem **1.1** and $P(z) = z^2 p(z)$. For a given $\delta > 0$ there exists $R > 0$ such that $J_P(\delta) \subset D(0;R)$. Consider $\{S_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$ as in Remark **3.5**. Let V_R^+, V_R^- and V_R be defined as:

$$V_R = \{z \in \mathbb{C}^k : |z_i| \leq R \ \text{ for all } \ 1 \leq i \leq k\}$$

$$V_R^+ = V_R^1 \quad \text{and} \quad V_R^- = \bigcup_{i=2}^{k} V_R^i$$

where for a fixed i, $1 \leq i \leq k$

$$V_R^i = \{z \in \mathbb{C}^k : |z_j| \leq \max\{|z_i|, R\} \ \text{ for every } \ 1 \leq j \leq k\}.$$

Lemma 4.2. *For $R > 0$, sufficiently large*

(i) *If $z \in \mathbb{C}^k$, $F(n)(z) \in V_R \cup V_R^+$.*
(ii) *If $z \in V_R^+$, $S(n)(z) \to \infty$.*

Proof. The arguments are same as in the proof of Lemma **3.2** and Lemma **3.3** in [5]. □

Let

$$N_C = \{(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k : z_1 \in \mathbb{C}, |z_i| < C \text{ for all } 2 \leq i \leq k\}$$

and $U \subset N_C$ be defined as:

$$U = \{(z_1, z_2, \ldots, z_k) \in N_C : P(z_1) \in J_P(\delta)\}. \tag{4.1}$$

Corresponding to the sequence $\{S_n\}$, let $K^+_{\{S_n\}}$ and $J^+_{\{S_n\}}$ denote the following sets:

$$K^+_{\{S_n\}} = \{z \in \mathbb{C}^k : \|S(n)(z)\| \text{ is bounded for every } n \geq 0\}, \; J^+_{\{S_n\}} = \partial K^+_{\{S_n\}}.$$

Lemma 4.3. *There exists a sequence $\{c_n\}$ of positive real numbers decreasing to zero such that if $|a_n| < \min\{|a_{n-1}|^2, c_n\}$ for every $n \geq 0$, then $J^+_{\{S_n\}} \cap N_C \subset U$.*

Proof. By Lemma **4.1**, there exists a sequence $\{c'_n\}$. Choose $\{c_n\}$ such that $0 < c'_n R < c_n$.

If z is in the compact component of $N_C \backslash U$. Then by the choice of c_n's it follows that $S_1(z)$ is in the compact component of $N_C \backslash U$. Further repetitive arguments using Lemma **4.1**, shows that $\pi_1 \circ S(n)(z) \to 0$. Also, $\pi_i \circ S(n+i) = a_{n+i}\pi_1 \circ S(n)(z)$ for $2 \leq i \leq k$. Hence it follows that $S(n)(z) \to 0$ as $n \to \infty$.

If z is in the non–compact component of $N_C \backslash U$, then there are two cases.

Case 1. If $|\pi_i \circ S(n)(z)| \leq R$ for every $2 \leq i \leq R$ and $n \geq 0$, then the choice of a_n's and Lemma **4.1** assures that $\pi_1 \circ S(n)(z) \to \infty$ as $n \to \infty$.

Case 2. Otherwise suppose $|\pi_{i_0} \circ S(\tilde{n})(z)| > R$ for some $\tilde{n} \geq 1$ and $2 \leq i_0 \leq k$. Also let $|\pi_i \circ S(n)(z)| \leq R$ for every $0 \leq n < \tilde{n}$ and $2 \leq i \leq k$. If $i_0 > 2$, then $|\pi_{i_0-1} S(\tilde{n}-1)| > R$ which contradicts the choice of $\tilde{n}$, i.e., $i_0 = 2$. Since, $|\pi_i \circ S(\tilde{n}-1)(z)| < R$ for every $2 \leq i \leq k$ and $|\pi_1 \circ S(\tilde{n}-1)(z)| > R$ it follows that $S(\tilde{n}-1)(z) \in V_R^+$, i.e., $S(n)(z) \to \infty$ as $n \to \infty$.

□

The proof of Theorem **5.1** in [9] relied on the following idea:

> 'The Fatou–Bieberbach domains $F(n-1)(\Omega^{a_n})$'s constructed for every $n \geq 0$ were converging to $\Omega_{\{F_n\}}$ in the Hausdorff metric on sufficiently large polydiscs in $\mathbb{C}^2$.'

However, the proof of Theorem **4.4** does not use this idea. On the contrary, it involves the convergence of forward Julia sets of a sequence of automorphisms to a standard object whose Hausdorff dimension is predetermined. Let us recall a few definitions and standard notations before proceeding to the result:

Let K be a compact subset of some metric space, say X. For $\epsilon > 0$ let $\mathcal{B}_\epsilon$ denote the collection of all coverings of K by balls of radius ϵ, i.e.,

$$\mathcal{B}_\epsilon = \left\{ \{B_i\} : K \subset \bigcup_i B_i \quad \text{and} \quad B_i = B(p_i; \epsilon) \text{ for some } p_i \in X \right\}.$$

For $h \geq 0$ define

$$\gamma_h^\epsilon(K) = \epsilon^h \inf_{\mathcal{B}_\epsilon} \#\{B_i\} \quad \text{and} \quad \mu_h(K) = \limsup_{\epsilon \to 0} \gamma_k^\epsilon(K).$$

$\mu_h(K)$ is called the h-upper–box content (or the Minkowski content) of K. The upper–box dimension of K is denoted by $\overline{\dim}_B(K)$ and is defined as the unique value of $h \geq 0$ such that

$$\mu_{h'}(K) = \begin{cases} 0 & \text{for every } \ h' > h \quad \textit{and} \\ \infty & \text{for every } \ h < h'. \end{cases}$$

The upper–box dimension of the subset K is always greater than or equal to the Hausdorff dimension (see [4]).

For two compact sets $A, B \subset \mathbb{C}^k$, the definition of Hausdorff distance between A and B is given by

$$d_H(A, B) = \max\{d(A, B), d(B, A)\}$$

where

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

Theorem 4.4. *There exists an $\alpha_0 > 1$ and a Short $\mathbb{C}^k$, say Ω such that the upper–box dimension is greater than or equal to $2(k-1) + \alpha_0$ at every point in the boundary of Ω. Further, Ω is obtained as a non–autonomous basin of attraction of a sequence of automorphisms in $\mathbb{C}^k$.*

Proof. Note that if $a > 0$ and $b > 0$ are chosen appropriately the polynomial $p(z) = az^4 + bz^3 + z^2$ satisfies the following properties:

(i) $p(z)$ is a hyperbolic polynomial with a single attracting cycle only at the origin. This is possible since z^2 is hyperbolic with only one component and degree 4 hyperbolic polynomials form an open subset in the space degree 4 polynomials.
(ii) By Theorem **4.4.20** in [8], it follows that Fatou set of $p(z)$ has only two connected components, i.e., the component containing the origin and the component containing infinity. Further, the Julia set of P is the boundary of the Fatou component containing the origin.
(iii) The Hausdorff dimension of $J(p) = \alpha_0 > 1$. This follows from Theorem **1.4.2** in [8].

Choose $C > 0$ sufficiently small and let N_C (as before) be a C-neighbourhood of the z_1-axis in $\mathbb{C}^k$. From the proof of Lemma **4.3**, for some $\delta > 0$ there exists a positive sequence $\{a_n(\delta)\}$ such that for $S_n = S_{a_n(\delta)}$,

$$J^+_{\{S_{a_n}\}} \cap N_C \subset J_p(\delta) \times D^{k-1}(0; C).$$

Let $\mathcal{J} = J_p \times D^{k-1}(0; C)$. Then the Hausdorff dimension of $\mathcal{J}$ is equal to $2(k-1) + \alpha_0$. Let h_n be a sequence increasing to $2(k-1) + \alpha_0$. The final sequence S_n will be constructed inductively.

Induction hypothesis: There exist $(i+1)$-constants $\{a_j \in \mathbb{R}^+ : 0 \leq j \leq i\}$ such that $S_j = S_{a_j}$, $0 \leq j \leq i$ satisfies the following properties:

- There exists a finite collection of balls $\mathcal{B}_i$ of radius $2^{-(i+1)}$ covering $\mathcal{J}_i = S(i-1)^{-1}(\mathcal{J})$ such that every element of $\mathcal{B}_i$ intersect $\mathcal{J}_i$. Further, there exists $\hat{\epsilon}_i > 0$ such that $\gamma_{h_i}^{\hat{\epsilon}_i}(\mathcal{J}_i \cap B) > 2^{i+1}$ for every $B \in \mathcal{B}_i$.
- Let $0 < \eta_i < \hat{\epsilon}_i - \epsilon_i$, where $2\epsilon_i^{h_i} = \hat{\epsilon}_i^{h_i}$. There exists a sequence of positive real numbers $\{a_k^i\}$ such that the finite collection $\{S_j : 0 \leq j \leq i\}$ is completed with $S_{i+k} = S_{a_{i+k}}$ for $k \geq 1$ where $a_{i+k} \leq \max\{a^3_{i+k-1}, a^i_k\}$ then

$$d_H(J^+_{\{S_n\}} \cap S(i-1)^{-1}(N_C), \mathcal{J}_i) < \eta_i.$$

Initial step: When $i = 0$, consider a covering of $\mathcal{J}$ by balls of radius $1/2$, say $\mathcal{B}_0$ such that every element of $\mathcal{B}_0$ intersect $\mathcal{J}$. Further, let $\hat{\epsilon}_0$ be such that $\gamma_{h_0}^{\hat{\epsilon}_0}(\mathcal{J} \cap B) > 2$ for every $B \in \mathcal{B}_0$. Let $2\epsilon_0^{h_0} = \hat{\epsilon}_0^{h_0}$ and $0 < \eta_0 < \hat{\epsilon}_0 - \epsilon_0$. Also consider $U_0 = J_p(\eta_0) \times D^{k-1}(0; C)$. Then by Lemma **4.3**, there exists a sequence of positive real numbers $\{a_k^0\}$ such that if $S_k = S_{a_k}$ where $a_k \leq \max\{a^3_{k-1}, a^0_k\}$ for every $k \geq 0$ then $\Omega_{\{S_n\}}$ is a *Short* $\mathbb{C}^k$ and $J^+_{\{S_n\}} \cap N_C \subset U_0$. Let $S_0 = S_{a_0}$.

General step: Suppose the statement is true for some $i \geq 0$. Consider a covering of $\mathcal{J}_{i+1} = S(i)^{-1}(\mathcal{J})$ by balls of radius 2^{i+2} , say $\mathcal{B}_{i+1}$ such that every element of $\mathcal{B}_{i+1}$ intersects $\mathcal{J}_{i+1}$. Further, $\hat{\epsilon}_{i+1}$ such that $\gamma_{h_{i+1}}^{\hat{\epsilon}_{i+1}}(\mathcal{J} \cap B) > 2^{i+2}$

for every $B \in \mathcal{B}_{i+1}$. Let $2\epsilon_{i+1}^{h_{i+1}} = \hat{\epsilon}_{i+1}^{h_{i+1}}$ and $0 < \eta_{i+1} < \hat{\epsilon}_{i+1} - \epsilon_{i+1}$. Further, choose $0 < \tilde{\eta}_{i+1} < \eta_0$ such that for $z, w \in U_0$

$$\|S(i)^{-1}(z) - S(i)^{-1}(w)\| < \eta_{i+1} \text{ whenever } \|z - w\| < \tilde{\eta}_{i+1}.$$

Let $U_{i+1} = J_p(\tilde{\eta}_{i+1}) \times D^{k-1}(0; C)$. Then by Lemma **4.3**, there exists a sequence of positive real numbers $\{a_k^{i+1}\}$ such that if $S_k = S_{a_k}$ where $a_k \leq \max\{a_{k-1}^3, a_{k-i}^{i+1}\}$ for every $k \geq i+1$ such that the $\Omega_{\{S_n\}}$ is a *Short* $\mathbb{C}^k$ and $S(i)(J^+_{\{S_n\}}) \cap N_C \subset U_{i+1}$, i.e.,

$$d_H(J^+_{\{S_n\}} \cap S(i)^{-1}(N_C), \mathcal{J}_{i+1}) < \eta_{i+1}.$$

Let $S_{i+1} = S_{a_{i+1}}$.

Hence it is possible to obtain $\{S_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$ such that for every $n \geq 0$,

- There exists $\mathcal{B}_n$ a finite collection of balls of radius $2^{-(n+1)}$ covering $\mathcal{J}_n = S(n-1)^{-1}(\mathcal{J})$ such that every element of $\mathcal{B}_n$ intersect $\mathcal{J}_n$. Further, there exists $\hat{\epsilon}_n > 0$ such that $\gamma_{h_n}^{\hat{\epsilon}_n}(\mathcal{J}_i \cap B) > 2^{n+1}$ for every $B \in \mathcal{B}_n$.
- There exists $0 < \eta_n < \hat{\epsilon}_n - \epsilon_n$, where $2\epsilon_n^{h_n} = \hat{\epsilon}_n^{h_n}$ such that

$$d_H(J^+_{\{S_n\}} \cap S(n-1)^{-1}(N_C), \mathcal{J}_n) < \eta_n.$$

Let $z \in J^+_{\{S_n\}}$. Then for sufficiently large $n \geq n_z \geq 0$, $z \in S(n)^{-1}(N_C)$. Choose $\epsilon > 3.2^{-(n+2)}$. Let $w \in \mathcal{J}_{n+1}$ such that $z \in B^k(w; \eta_{n+1})$. By assumption $\mathcal{B}_{n+1}$ is a covering by $2^{-(n+2)}$ balls of $\mathcal{J}_{n+1}$. Let B_w be the ball in $\mathcal{B}_{n+1}$ that contains w, then $B_w \in B^k(z; \epsilon)$. Consider any arbitrary covering $\{\tilde{B}_j\}$ of $J^+_{\{S_n\}} \cap B^k(z; \epsilon)$ by balls of radius ϵ_{n+1}. Further, let $\{B'_j\}$ represent the collection of balls with same centers as $\tilde{B}_j$ but radius $\hat{\epsilon}_{n+1}$. Since $\eta_{n+1} < \hat{\epsilon}_{n+1} - \epsilon_{n+1}$ and

$$d_H(J^+_{\{S_n\}} \cap S(n)^{-1}(N_C), \mathcal{J}_{n+1}) < \eta_{n+1}$$

$\{B'_j\}$ is a covering of $\mathcal{J}_{n+1} \cap B_w$. Further, $\hat{\epsilon}_{n+1}^{h_{n+1}} \#\{B'_j\} > 2^{n+2}$ for every $n \geq n_z$. Now let $h < 2(k-1) + \alpha_0$. Then for sufficiently large n, $h_n \geq h$

$$\gamma_h^{\epsilon_{n+1}}(J^+_{\{S_n\}} \cap B^k(z; \epsilon)) > \gamma_{h_{n+1}}^{\epsilon_{n+1}}(J^+_{\{S_n\}} \cap B^k(z; \epsilon)) > 2^{n+1}. \tag{4.2}$$

Since (4.2) is true for all n, sufficiently large it follows that $\mu_h(J^+_{\{S_n\}} \cap B^k(z; \epsilon)) = \infty$, i.e., the box dimension at z is greater than h. Hence the upper–box dimension of $J^+_{\{S_n\}}$ at every point is greater than or equal to $2(k-1)+\alpha_0$. □

Remark 4.5. By Theorem **6.1** in [14], for $\delta > 0$ there exists $a_0(\delta) > 0$ such that the forward Julia set (J_a^+) of the automorphism

$$H_a(z_1, z_2) = (a^2 z_2 + p(z_1), z_1)$$

has Hausdorff dimension $h_a \in (2 + \alpha_0 - \delta, 2 + \alpha_0 + \delta)$ whenever $0 < |a| < a_0(\delta)$. Since

$$S_a = \mathcal{L}_a \circ H_a \circ \mathcal{L}_{a^{-1}} = (aw + p(z), az)$$

where $\mathcal{L}_a(z_1, z_2) = (z_1, az_2)$. So the Hausdorff dimension of the forward Julia set of S_a is h_a. Let Ω^a denote the attracting basin of attraction of S_a. From [1] it follows that $J_a^+ = \partial\Omega^a$. Theorem **4.4** says that

$$\overline{\dim}_H(J^+_{a_n}) \to \overline{\dim}_B(J^+_{\{F_n\}}).$$

Proposition 4.6. *Let $\{S_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$ be the sequence as constructed in the proof of Theorem **4.4**. Then $K^+_{\{S_n\}}$ is connected and*

$$K^+_{\{S_n\}} = \overline{\Omega_{\{S_n\}}} \quad \textit{and} \quad J^+_{\{S_n\}} = \partial\Omega_{\{S_n\}}.$$

Proof. Choose $z_0 \in J^+_{\{S_n\}}$. Since $z_0 \in K^+_{\{S_n\}}$ and $\eta_n \to 0$ as $n \to \infty$, there exists $n_0 \geq 0$, sufficiently large such that

$$S(n_0)(z_0) \in N_{C-\tilde{\eta}_{n_0}} \quad \text{and} \quad \eta_{n_0} < \epsilon/3.$$

Claim. For $z \in J_p \times D^{k-1}(0; C-\tilde{\eta}_{n_0})$ and $r > \tilde{\eta}_{n_0}$ there exists θ_1^z and θ_2^z in $B^k(z;r)$ such that $S(n)S(n_0)^{-1}(\theta_1^z) \to 0$ and $S(n)S(n_0)^{-1}(\theta_2^z) \to \infty$ as $n \to \infty$.

Recall that $U_{n_0} = J_p(\tilde{\eta}_{n_0}) \times D^{k-1}(0; C)$. Hence,

$$B^k(z; \tilde{\eta}_{n_0}) \subset U_{n_0} \ \text{ for } \ z \in J_p \times D^{k-1}(0; C - \tilde{\eta}_{n_0}).$$

Let $z = (z_1, z')$ where $z_1 \in J_p$ and $z' \in D^{k-1}(0; C - \tilde{\eta}_{n_0})$. Now for $r > r_0 > \tilde{\eta}_{n_0}$ consider the points $\theta_t = (z_1 + r_0 e^{it}, z')$ for $t \in [0, 2\pi]$. Then $\theta_t \in B^k(z;r) \backslash B^k(z; \tilde{\eta}_{n_0})$ for every t. Further, there exists t_1 and t_2 such that $z_1 + r_0 e^{it_1}$ lies in the compact component of $\mathbb{C} \backslash J_p(\tilde{\eta}_{n_0})$ and $z_1 + r_0 e^{it_2}$ lies in the non–compact component respectively. Thus $\theta_1^z = \theta_{t_1}$ lies in the compact component of $N_C \backslash U_{n_0}$ and $\theta_2^z = \theta_{t_2}$ in the non–compact component. By the property of $\{S_n\}$'s, it follows that $S(n)S(n_0)^{-1}(\theta_1^z) \to 0$ and $S(n)S(n_0)^{-1}(\theta_2^z) \to \infty$ as $n \to \infty$.

Observe that $S(n_0)(z_0) \in N_{C-\tilde{\eta}_{n_0}} \cap U_{n_0}$, i.e., there exists $\tilde{z} \in J_p \times D^{k-1}(0; C - \tilde{\eta}_{n_0})$ such that

$$\|S(n_0)(z_0) - \tilde{z}\| < \tilde{\eta}_{n_0}.$$

Thus

$$\|z_0 - S(n_0)^{-1}(\tilde{z})\| < \eta_{n_0}$$

and $S(n_0)^{-1}(\tilde{z}) \in B^k(z_0; \epsilon)$. Also by the choice η_{n_0}, it follows $B^k(S(n_0)^{-1}(\tilde{z}); \eta_{n_0}) \subset B^k(z_0; \epsilon)$. Now

$$\overline{B^k(\tilde{z}; \tilde{\eta}_{n_0})} \subset S(n_0)(B^k(S(n_0)^{-1}(\tilde{z}); \eta_{n_0})),$$

i.e., there exists $r > \tilde{\eta}_{n_0}$ such that

$$B^k(\tilde{z}; r) \subset S(n_0)(B^k(S(n_0)^{-1}(\tilde{z}); \eta_{n_0})).$$

Thus from the above claim, there exist $s_1 = S(n_0)^{-1}(\theta_1^{\tilde{z}})$ and $s_2 = S(n_0)^{-1}(\theta_2^{\tilde{z}}) \in B^k(z;\epsilon)$ such that $S(n)(s_1) \to 0$ and $S(n)(s_2) \to \infty$ as $n \to \infty$. Since this is true for any arbitrary $\epsilon > 0$, it follows that $z \in \partial\Omega_{\{S_n\}}$. Thus the proof. □

5. Proof of Results 1.3–1.6

In this section, we prove some properties of biholomorphic images of non–autonomous basins of attraction at a fixed point that satisfy the *uniform upper–bound* condition. We assume that the non–autonomous basin of attraction is not all of $\mathbb{C}^k$, as in this case it is enough to show existence of Fatou–Bieberbach domains with these properties. Henceforth, we will assume that the non–autonomous basin of attraction is always a proper subset of $\mathbb{C}^k$. Recall the following result from [9]. We will also have occasions to use the facts stated in the remarks thereafter.

Theorem 5.1. *Let $K_1, K_2, \ldots, K_m$ be pairwise disjoint polynomially convex compact sets in $\mathbb{C}^k$ whose union is polynomially convex, and assume that $K_1, K_2, \ldots, K_l$ are star–shaped ($l \leq m$). Let $\phi_i \in \mathsf{Aut}(\mathbb{C}^k)$ be automorphisms for $1 \leq i \leq l$ so that the sets $K_i' = \phi_i(K_i)$ and the sets $K_{l+1}, \ldots, K_m$ are pairwise disjoint and their union is polynomially convex. Let $\epsilon > 0$. Then there exists an automorphism $\phi \in \mathsf{Aut}(\mathbb{C}^k)$ so that $\|\phi(z) - \phi_i(z)\| < \epsilon$ for all $z \in K_i$, $1 \leq i \leq l$ and $\|\phi(z) - z\| < \epsilon$ for all $z \in K_j$, $l+1 \leq j \leq m$.*

Remark 5.2.

(i) The union of a polynomially convex compact set and a finite set of points is polynomially convex.

(ii) If $K_1 \cup K_2$ is polynomially convex and compact, $K_1 \cap K_2 = \emptyset$, and $K_1' \subset K_1$ is polynomially convex and compact then $K_1' \cup K_2$ is polynomially convex.

(iii) A polynomially convex compact set has a neighbourhood basis consisting of polynomially convex compact sets.
(iv) The union of two disjoint polynomially convex compact set, that can be separated by two disjoint convex compact sets is polynomially convex.

Proof of Theorem 1.3. Since $\{S_n\}$ satisfies the condition of Theorem **1.2**, the sequence $\{\delta_n\}$ as in the proof of Theorem **1.2** gives a convergent series. So let $\epsilon_n = \sum_{i=0}^{n} \delta_i$ for every $n \geq 0$ and $\epsilon = \sum_{i=0}^{\infty} \delta_i$. Moreover, there exists $0 < r_0 < 1$ such that

$$\Omega_{\{S_n\}} = \bigcup_{i=0}^{\infty} \Omega_i^S \quad \text{where} \quad \Omega_i^S = S(i)^{-1}(B^k(0; r_0)).$$

Without loss of generality assume that there exists $p \in \mathbb{C}^k$ and $R > 0$ such that ϵ-neighbourhood of K, i.e., $K_\epsilon \subset \overline{B^k(p; R)}$ and $\overline{B^k(p; R)} \cap \overline{B^k(0; r_0 + \epsilon)} = \emptyset$. Let $\bar{B} = \overline{B^k(0; r_0)}$. Also let $p_0 = 0$.

Induction hypothesis: For every $i \geq 0$ there exist i-many automorphisms in $\mathsf{Aut}_0(\mathbb{C}^k)$ such that the following are true:

$$\|F_j - S_j\|_{\bar{B}} < \delta_j,$$

$$F(j)(p_j) \subset \bar{B} \quad \text{and} \quad F(j)(K) \subset K_{\epsilon_j} \subset \overline{B(p; R)} \subset \mathbb{C}^k \setminus \bar{B}$$

for every $0 \leq j \leq i$.

Initial step: By Remark **5.2**(iv) $\bar{B} \cup K$ is polynomially convex. Since, $S_0(\bar{B}) \subset \bar{B}$, $S_0(\bar{B}) \cup K$ is also polynomially convex. Hence, by Theorem **5.1**, for δ_0 there exists $\phi \in \mathsf{Aut}_0(\mathbb{C}^k)$ such that

$$\|\phi - S_0\|_{\bar{B}} < \delta_0 \quad \text{and} \quad \|\phi - \mathrm{Id}\|_K < \delta_0.$$

Let $F_0 = \phi$. Note that $\phi(K) \subset K_{\delta_0}$ and $\phi(p_0) \in \bar{B}$.

General step: Let $\mu_{i+1} = \delta_{i+1}/2$. By the same reasoning as before $\bar{B} \cup F(i)(K)$ is polynomially convex and $S_{i+1}(\bar{B}) \cup F(i)(K)$ is polynomially convex. Hence, by Theorem **5.1** there exists $\phi \in \mathsf{Aut}_0(\mathbb{C}^k)$ such that

$$\|\phi - S_{i+1}\|_{\bar{B}} < \mu_{i+1} \quad \text{and} \quad \|\phi - \mathrm{Id}\|_{F(i)(K)} < \mu_{i+1}.$$

From (3.2) in the proof of Theorem **1.2**, $\phi(\bar{B}) \subset \bar{B}$. Hence, $\phi \circ F(i)(p_j) \subset \bar{B}$ for every $0 \leq j \leq i$. Now if $F(i)(p_{i+1}) \in \phi^{-1}(\bar{B})$, then consider $F_{i+1} = \phi$.

Otherwise, if $F(i)(p_{i+1}) \notin \phi^{-1}(\bar{B})$, i.e., $F(i)(p_{i+1}) \notin \bar{B}$. From Remark **5.2**(i), $\bar{B} \cup F(i)(K) \cup F(i)(p_{i+1})$ is polynomially convex. Let $\tau_{i+1} \in \phi^{-1}(\bar{B}) \setminus (F(i)(K) \cup \bar{B})$, then $\bar{B} \cup F(i)(K) \cup \tau_{i+1}$ is also polynomially convex. There exists $1 > \rho > 0$ such that for $z, w \in (\bar{B} \cup F(i)(K))_1$, i.e., on a radius 1-neighbourhood of $\bar{B} \cup F(i)(K)$,

$$\|\phi(z) - \phi(w)\| < \mu_{i+1} \text{ whenever } \|z - w\| < \rho.$$

Hence, by Theorem **5.1** there exists $\psi \in \mathsf{Aut}_0(\mathbb{C}^k)$ such that

$$\|\psi - \mathrm{Id}\|_{\bar{B} \cup F(i)(K)} < \rho \quad \text{and} \quad \psi(F(i)(p_{i+1})) = \tau_{i+1} \in \phi^{-1}(\bar{B}).$$

Consider $F_{i+1} = \phi \circ \psi$. From the construction $F(i+1)(p_{i+1}) \in \bar{B}$. For $z \in \bar{B}$, then $\|\psi(z) - z\| < \rho$ and $\psi(z) \in \overline{B(0; 1 + r_0)}$. Thus by continuity of ϕ

$$\|\phi \circ \psi(z) - \phi(z)\| < \mu_{i+1} \quad \text{and} \quad \|\phi(z) - S_{i+1}(z)\| < \mu_{i+1},$$

i.e.,

$$\|F_{i+1}(z) - S_{i+1}(z)\| < \delta_{i+1}.$$

Similar arguments for $z \in F(i)(K)$ gives

$$\|F_{i+1}(z) - z\| < \delta_{i+1},$$

i.e., $F(i+1)(K) \subset (K_{\epsilon_i})_{\delta_{i+1}} = K_{\epsilon_{i+1}}$. Hence the induction statement is true for $i + 1$.

Now by Theorem **1.2**, $\Omega_{\{F_n\}}$ is biholomorphic to $\Omega_{\{S_n\}}$. Also $\{p_j\} \subset \Omega_{\{F_n\}}$ and $K \cap \Omega_{\{F_n\}} = \emptyset$. □

Proof of Corollary 1.4. Without loss of generality consider $p = 0$ and K sufficiently away from the origin. Let $\{p_j\}$ be a dense sequence in $\mathbb{C}^k \setminus K$. Then by Theorem **1.3**, there exists a sequence of automorphisms $\{F_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ such that $\Omega_{\{F_n\}}$ is biholomorphic to $\Omega_{\{S_n\}}$ and $\{p_j\} \subset \Omega_{\{F_n\}}$ and $\Omega_{\{F_n\}} \cap K = \emptyset$. But $\Omega_{\{F_n\}}$ is open and hence the proof. □

Corollary 5.3. *Given a sequence of automorphisms $\{S_n\} \in \mathsf{Aut}_0(\mathbb{C}^k)$ that satisfy the uniform upper–bound condition at the origin, there exists a biholomorphism of $\Omega_{\{S_n\}}$ (say Φ), such that the $2k$-dimensional Hausdorff measure of $\partial\Phi(\Omega_{\{S_n\}})$ is non–zero.*

Proof. Let $D = (\bar{D})^{\mathrm{o}} \subset\subset \mathbb{C}$ be a simply connected domain in $\mathbb{C}$ such that ∂D has non–zero two dimensional Hausdorff measure. Then $K = \overline{D^k} = \overline{D \times \cdots \times D} \subset \mathbb{C}^k$ is a polynomially convex compact set with non–zero $2k$-dimensional Hausdorff measure. By Corollary **1.4**, the result follows. □

Proof of Corollary 1.5. From Corollary **1.4**, for any given sequence $\{S_n\}$ there exists $\Phi_1(\Omega_{\{S_n\}}) \subset \mathbb{C}^* \times \mathbb{C}^{k-1}$. From Theorem **2.5**, there exists $\Phi_2 \in \mathsf{Aut}(\mathbb{C}^* \times \mathbb{C}^{k-1})$ such that $Y \subset \Phi_2^{-1} \circ \Phi_1(\Omega_{\{S_n\}})$. Let $\Phi = \Phi_2^{-1} \circ \Phi_1$. Then $\Phi(\Omega_{\{S_n\}})$ is not Runge. □

Proof of Theorem 1.6. Choose $\epsilon_n \to 0$ as $n \to \infty$.

Case 1. When $m = \infty$.

Induction hypothesis: For every $i \geq 0$, there exist

- $(i+1)$-automorphisms $\{F_j \in \mathsf{Aut}(\mathbb{C}^k) : 0 \leq j \leq i\}$,
- Two set of distinct points $P^i = \{p_j \in \mathbb{C}^k : 0 \leq j \leq i\}$ and $Q^i = \{q_j \in \mathbb{C}^k : 0 \leq j \leq i\}$,
- A set of positive numbers $\Gamma^i = \{\rho_j \in \mathbb{R}^+ : 0 \leq j \leq i\}$,

with the following properties:

(i) $\overline{B^k(q_j;\rho_j)} \cap \overline{B^k(q_k;\rho_k)} = \emptyset$ for $0 \leq j \neq k \leq i$.
(ii) $F(i)(p_j) = q_j$ for all $0 \leq j \leq i$.
(iii) $F_j(q_k) = q_k$ for all $0 \leq k \leq j$ and $0 \leq j \leq i$
(iv) For every $z \in \overline{B^k(q_k;\rho_k)}$

$$\|F_j(z) - S_j(z - q_k) - q_k\| < \delta_j(\rho_k)$$

whenever $0 \leq k \leq j$ and $0 \leq j \leq i$. Here $\delta_j(\rho_k)$ is as observed in Remark **3.7**.
(v) $B^i = \cup_{j=0}^i \overline{B^k(q_j;\rho_j)}$ is polynomially convex.
(vi) For $t \in B^k(0;i) \setminus F(i)^{-1}(B^i)$ and for every $j, 0 \leq j \leq i$

$$\mathrm{dist}\Big(t, F(i)^{-1}\big(\overline{B^k(q_j;\rho_j)}\big)\Big) < \epsilon_i.$$

(vii) $P^j \subset P^{j+1}$, $Q^j \subset Q^{j+1}$ and $\Gamma^j \subset \Gamma^{j+1}$ where $0 \leq j \leq i-1$.

Initial step: Let $p_0 = q_0 = 0$ be the origin, $F_0 = S_0$ and $\rho_0 = r$, as in Theorem **1.2**. Since $i = 0$, all the conditions are true.

General step: Suppose all the assumptions are true for some $i \geq 0$. Let $K_i = F(i)^{-1}(B^i)$, i.e., K_i is polynomially convex. For every $0 \leq j \leq i+1$ consider a set of points

$$T^j = \{t_l^j : 1 \leq l \leq m_j\} \subset B^k(0; i+1) \setminus int(K_i)$$

for some $m_j \geq 1$, such that $T^j \cap T^k = \emptyset$ whenever $0 \leq j \neq k \leq i+1$. Also for each $j, 0 \leq j \leq i+1$

$$\mathrm{dist}(t, T^j) < \epsilon_{i+1} \quad \text{whenever} \quad t \in B^k(0; i+1) \setminus int(K_i).$$

Let $\mathscr{T}^{i+1} = \cup_{j=0}^{i+1} T^j$ and $\tau^j = F(i)(T^j)$. Choose a point $p_{i+1} \in \mathbb{C}^k \setminus (\mathscr{T}^{i+1} \cup K_i)$ and let $q_{i+1} = F(i)(p_{i+1})$. Now there exists $\rho_{i+1} > 0$ such that

$$\overline{B^k(q_{i+1}; \rho_{i+1})} \cap \left(\bigcup_{j=0}^{\infty} \tau^j \cap B^i \right) = \emptyset.$$

Further, from Remark **5.2**(iii) we have the following:

- By appropriately modifying ρ_{i+1} we have that $\overline{B^k(q_{i+1}; \rho_{i+1})} \cup B^i$ is polynomially convex.
- There exists $\mu > 0$, $\cup_{j=0}^{i+1} \overline{B^k(q_j; \rho_j + \mu)}$ is polynomially convex and $\overline{B^k(q_j; \rho_j + \mu)} \cap \overline{B^k(q_k; \rho_k + \mu)} = \emptyset$ whenever $0 \leq j \neq k \leq i+1$.
- Let $\psi_j(z) = S_{i+1}(z - q_j) + q_j$. Then $\cup_{j=0}^{i+1} \psi_j\big(\overline{B^k(q_j; \rho_j + \mu)}\big)$ is polynomially convex.

By Theorem **5.1** there exists $\phi \in \mathsf{Aut}(\mathbb{C}^k)$ such that for every $0 \leq j \leq i+1$

$$\|\phi(z) - \psi_j(z)\| < \mu_{i+1}$$

where $\mu_{i+1} = \min\{\mu, \delta_{i+1}(\rho_j)/2 : 0 \leq j \leq i+1\}$ and $\phi(q_j) = q_j$. By continuity of ϕ, there exists $\tilde{\mu}_{i+1} < \mu_{i+1}$ such that on $\cup_{j=0}^{i+1} \psi_j\big(\overline{B^k(q_j; \rho_j + \mu)}\big)$

$$\|\phi(z) - \phi(w)\| < \mu_{i+1} \quad \text{whenever} \quad \|z - w\| < \tilde{\mu}_{i+1}. \tag{5.1}$$

Again, by Theorem **5.1** there exists $\psi \in \mathsf{Aut}(\mathbb{C}^k)$ such that

$$\|\psi(z) - z\| < \tilde{\mu}_{i+1}$$

on each $\overline{B^k(q_j, \rho_j)}$ and $\psi(\tau^j) \subset \phi^{-1}(B^k(q_j, \rho_j))$ for every $0 \leq j \leq i+1$. Further, $\psi(q_j) = q_j$. Let $F_{i+1} = \phi \circ \psi$.

Clearly, the collection $\{F_j : 0 \leq j \leq i+1\}$ satisfies all the properties (i)–(iii), (v) and (vii). Let $z \in \overline{(B^k(q_j, \rho_j))}$, then $\psi(z) \in \overline{B^k(q_j, \rho_j + \mu)}$. From (5.1)

$$\|F_{i+1}(z) - \phi(z)\| < \mu_{i+1}, \quad \text{i.e.,} \quad \|F_{i+1}(z) - \psi_j(z)\| < \delta_{i+1}(\rho_j)$$

for every $0 \leq j \leq i+1$. Hence property (iv) is true.

Also, $F(i+1)(T^j) \subset B^k(q_j, \rho_j)$ for every $0 \leq j \leq i+1$ and by choice of T_j's property (vi) is also satisfied.

Let $\{S_n^i\}$ denote the sequence $S_n^i = S_{i+n}$ for every $i \geq 0$. Now from the sequence $\{F_n\}$ obtained the non–autonomous basin of attraction at every point q_i, i.e., $\Omega_{\{F_n^i\}} \cong \Omega_{\{S_n^i\}}$ for $i \geq 0$. Since $\Omega_{\{S_n^i\}} = S(i)(\Omega_{\{S_n\}})$, it follows that $\Omega_{\{F_n^i\}} \cong \Omega_{\{S_n\}}$. Now by construction $\Omega_{\{F_n^i\}} \cap \Omega_{\{F_n^j\}} = \emptyset$ for $i \neq j$. Also for any given $\epsilon > 0$, there exists $n_0 \geq 0$ such that $\epsilon_{n_0} < \epsilon$, hence for every $i \geq 0$ and $t \notin \mathbb{C}^k \setminus \cup_{i=0}^{\infty} \Omega_{\{F_n^i\}}$

$$\text{dist}(t, \partial\Omega_{\{F_n^i\}}) < \epsilon.$$

Thus $t \in \partial\Omega_{\{F_n^i\}}$ for every $i \geq 0$.

Case 2. When $m < \infty$.

For $p_{m+i} = q_m$ for every $i \geq 1$ and follow the same procedure as for the infinite case. □

6. Proof of Theorem 1.7

In this section we use Theorem **1.2** to prove that there exists biholomorphic images of non–autonomous basins of attraction at a point satisfying the *uniform upper–bound* condition with completely chaotic boundary. The technique is adapted from Theorem **1.1** from [9].

Proof of Theorem 1.7. Let $D = int(\bar{D})$ be a simply connected domain in $\mathbb{C}$ such that the Hausdorff dimension of ∂D is 2. Let $K = D^k = D \times D \cdots \times D$, then the Hausdorff dimension of ∂K is $2k$. Also for any $p \in \mathbb{C}^K$ and $\epsilon > 0$ there exists an appropriate affine transformation $\phi_{p,\epsilon}$ such that $p \in \phi_{p,\epsilon}(K) \subset B^k(p; \epsilon)$. Let $K(p; \epsilon) = \phi_{p,\epsilon}(K)$. Let $r > 0$ and $\{\delta_n\}$ be as obtained in Theorem **1.2**. Further, let

$$\tilde{\delta}_n = \sum_{j=n}^{\infty} \delta_j.$$

Choose $\epsilon_n \to 0$ as $n \to \infty$.

Induction hypothesis: For every $i \geq 0$, there exist

- $(i+1)$-automorphisms $\{F_j \in \mathsf{Aut}_0(\mathbb{C}^k) : 0 \leq j \leq i\}$,
- Three set of distinct points $P^i = \{p^i_j \in \mathbb{C}^k : 0 \leq j \leq n(i)\}$, $Q^i = \{q_j : 0 \leq j \leq i\}$ and $T^i = \{t^i_j \in \mathbb{C}^k : 0 \leq j \leq m(i)\}$, where $m(i), n(i) > 0$ for every $i \geq 0$
- Two set of positive numbers $\Gamma^i = \{\rho_j \in \mathbb{R}^+ : 0 \leq j \leq i\}$ and $R^i = \{R_j \in \mathbb{R}^+ : 0 \leq j \leq i\}$

with the following properties:

(i) $\|F_i - S_i\| < \delta_i$ on $\bar{B}$.
(ii) $B^k(0; i+1) \backslash F(i)(\bar{B}) \neq \phi$.
(iii) $F(i)(T^i) \in \bar{B}$.
(iv) $K^i = F(i-1)^{-1}\big(\cup_{j=0}^{n(i)} K(p^i_j; \rho_j)\big) \cup K^{i-1}$ is polynomially convex.
(v) For every $p \in B^k(0; i+1) \backslash int(K^i)$, $\text{dist}(p, T^i) < \epsilon_i$.
(vi) For every $p \in \overline{B^k(0; i+1) \backslash F(i)^{-1}(\bar{B})}$, $\text{dist}(p, K^i) < \epsilon_i$.
(vii) $\bar{B} \cap \overline{B^k(q_i; R_i)} = \emptyset$, $\text{dist}(\bar{B}, \overline{B^k(q_i; R_i)}) > \tilde{\delta}_i$ and $B^k(q_j; R_j) \subset B^k(q_{j+1}; R_{j+1})$ for every $0 \leq j \leq i-1$.
(viii) $R^j \subset R^{j+1}$, $Q^j \subset Q^{j+1}$ and $\Gamma^j \subset \Gamma^{j+1}$ where $0 \leq j \leq i-1$.
(ix) $F(i)(K^i) \subset B^k(q_i; R_i)$.

Initial step: Let $P^0 = \{p^0_j \in B^k(0; 1) \backslash \bar{B} : 1 \leq j \leq n(0)\}$ for some $n(0) \geq 1$ such that for any point in

$$p \in \overline{B^k(0; 1)} \backslash B, \ \text{dist}(p, P^0) < \epsilon_0.$$

Further, from Remark **5.2** there exists $\rho_0 > 0$ such that the following are true:

(a) $\bar{B} \cup \{\overline{B^k(p^0_j, \rho_0)} : 1 \leq j \leq n(0)\}$ is polynomially convex.
(b) $\overline{B^k(p^0_j, \rho_0)} \cap \overline{B^k(p^0_l, \rho_0)} \neq \emptyset$ for $1 \leq j \neq l \leq n(0)$ and

$$B^0 = \bigcup_{j=1}^{n(0)} \overline{B^k(p^0_j, \rho_0)} \ \text{is polynomially convex.}$$

Let

$$K^0 = \bigcup_{j=1}^{n(0)} K(p^0_j, \rho_0),$$

then again from Remark **5.2**, it follows that K^0 and $\bar{B} \cup K^0$ is polynomially convex. Let $T^0 = \{t^0_j \in B^k(0; 1) \backslash K^0 : 1 \leq j \leq m(0)\}$ for some $m(0) \geq 1$ be a collection of points such that for every

$$p \in \overline{B^k(0; 1)} \backslash int(K^0), \quad \text{dist}(p, T^0) < \epsilon_0.$$

Choose $q_0 \in \mathbb{C}$ and $R_0 > 0$, sufficiently large such that $\text{dist}(\bar{B}, \overline{B^k(q_0; R_0)}) > \tilde{\delta}_0$. Since, $\bar{B} \cup \overline{B^k(q_0; R_0)}$ is polynomially convex and $S_0(\bar{B}) \subset B$, from Remark **5.2**(ii) it follows that $S_0(\bar{B}) \cup \overline{B^k(q_0; R_0)}$ are polynomially convex. Further, let $0 < \mu_0 < \delta_0/2$ be chosen appropriately such that

$$\|S_0(z) - S_0(w)\| < \delta_0/2 \quad \text{whenever} \quad \|z - w\| < \mu_0$$

for every $z \in \overline{B^k(0; 1 + \delta_0)}$. Hence from Theorem **5.1**, there exists $\phi_1, \phi_2, \phi_3 \in \mathsf{Aut}_0(\mathbb{C}^k)$ such that

$$\|\phi_1 - S_0\|_{\bar{B}_{\delta_0}} < \delta_0/2 \quad \text{and} \quad \|\phi_1 - \text{Id}\|_{\overline{B^k(q_0; R_0)}} < \delta_0/2, \tag{6.1}$$

$$\|\phi_2 - \text{Id}\|_{\bar{B}} < \mu_0/2 \quad \text{and} \quad \phi_2(B^0) \in B^k(q_0; R_0 - \delta_0) \text{ and} \tag{6.2}$$

$$\|\phi_3 - \text{Id}\|_{\phi_2(\bar{B}) \cup \phi_2(K^0)} < \mu_0/2 \quad \text{and} \quad \phi_3 \circ \phi_2(T^0) \in \phi_1^{-1}(B). \tag{6.3}$$

Claim. $F_0 = \phi_1 \circ \phi_3 \circ \phi_2$ satisfies the induction hypothesis for $i = 0$.

Note that by choice F_0 satisfies properties (ii)–(v) and (vii)–(ix). Let $z \in \bar{B}$

$$\|\phi_3 \circ \phi_2(z) - z\| \leq \|\phi_3 \circ \phi_2(z) - \phi_2(z)\| + \|\phi_2(z) - z\| < \mu_0.$$

Also $\phi_3 \circ \phi_2(\bar{B}) \subset B_{\delta_0}$ and from the choice of μ_0 it follows that

$$\|S_0 \circ \phi_3 \circ \phi_2(z) - S_0(z)\| \leq \delta_0/2.$$

Thus

$$\|\phi_1 \circ \phi_3 \circ \phi_2(z) - S_0 \circ \phi_3 \circ \phi_2(z)\| < \delta_0/2.$$

Hence $\|F_0 - S_0\|_{\bar{B}} < \delta_0$, i.e., (i) is true. Also from relation (3.2) in the proof of Theorem **1.2**, $\bar{B} \subset F_0^{-1}(\bar{B})$, i.e., (vi) is also true.

Induction step: Suppose the conditions are true for some $i \geq 0$. Let

$$\tilde{P}^{i+1} = \{\tilde{p}_j^{i+1} \in B^k(0; i+2) \setminus \big(K^i \cup F(i)^{-1}(\bar{B})\big) : 1 \leq j \leq n(i+1)\}$$

for some $n(i+1) \geq 1$ such that for any point in

$$p \in \overline{B^k(0; i+2)} \setminus \big(int(K^i) \cup F(i)^{-1}(B)\big), \ \text{dist}(p, \tilde{P}^{i+1}) < \epsilon_{i+1}.$$

Let $P^{i+1} = F(i)(\tilde{P}^{i+1})$ and $p_j^{i+1} = F(i)(\tilde{p}_j^{i+1})$ for $1 \leq j \leq n(i+1)$. Further, from Remark **5.2** there exists $\rho_{i+1} > 0$ such that the following are true:

(a) $\bar{B} \cup F(i)(K^i) \cup \{\overline{B^k(p_j^{i+1}, \rho_{i+1})} : 1 \leq j \leq n(i+1)\}$ is polynomially convex.

(b) $\overline{B^k(p_j^{i+1}, \rho_{i+1})} \cap \overline{B^k(p_l^{i+1}, \rho_{i+1})} \neq \emptyset$ for $1 \leq j \neq l \leq n(i+1)$ and

$$B^{i+1} = F(i)(K^i) \cup \left(\bigcup_{j=1}^{n(i+1)} \overline{B^k(p_j^{i+1}, \rho_{i+1})} \right) \text{ is polynomially convex.}$$

Let

$$K^{i+1} = K^i \cup F(i)^{-1} \left(\bigcup_{j=1}^{n(i+1)} K(p_j^{i+1}, \rho_{i+1}) \right),$$

then again from Remark **5.2**, it follows that K^{i+1} and $F(i)^{-1}(\bar{B}) \cup K^{i+1}$ is polynomially convex. Let $T^{i+1} = \{t_j^{i+1} \in B^k(0; i+2)\backslash K^{i+1} : 1 \le j \le m(i+1)\}$ for some $m(i+1) \ge 1$ be a collection of points such that for every

$$p \in \overline{B^k(0; i+2)}\backslash int(K^{i+1}), \quad \text{dist}(p, T^{i+1}) < \epsilon_{i+1}.$$

Choose $q_{i+1} \in \mathbb{C}$ and $R_{i+1} > 0$, sufficiently large such that $\text{dist}(\bar{B}, \overline{B^k(q_{i+1}; R_{i+1})}) > \tilde{\delta}_{i+1}$ and $B(q_i; R_i) \subset B(q_{i+1}; R_{i+1} - \delta_{i+1})$. Since, $\bar{B} \cup \overline{B^k(q_{i+1}; R_{i+1})}$ is polynomially convex and $S_0(\bar{B}) \subset B$, from Remark **5.2**(ii) it follows that $S_{i+1}(\bar{B}) \cup \overline{B^k(q_{i+1}; R_{i+1})}$ is polynomially convex. Let $0 < \mu_{i+1} < \delta_{i+1}/2$ be chosen appropriately such that

$$\|S_i(z) - S_i(w)\| < \delta_{i+1}/2 \quad \text{whenever} \quad \|z - w\| < \mu_{i+1}$$

for every $z \in \overline{B^k(0; i+2+\delta_{i+1})}$. Hence from Theorem **5.1**, there exists $\phi_1, \phi_2, \phi_3 \in \mathsf{Aut}_0(\mathbb{C}^k)$ such that

$$\|\phi_1 - S_{i+1}\|_{\bar{B}_{\delta_{i+1}}} < \delta_{i+1}/2 \quad \text{and} \quad \|\phi_1 - \text{Id}\|_{\overline{B^k(q_{i+1}; R_{i+1})}} < \delta_{i+1}/2, \tag{6.4}$$

$$\|\phi_2 - \text{Id}\|_{\bar{B}} < \mu_{i+1}/2 \quad \text{and} \quad \phi_2(B^{i+1}) \in B^k(q_{i+1}; R_{i+1} - \delta_{i+1} + \mu_{i+1}/2) \text{ and} \tag{6.5}$$

$$\|\phi_3 - \text{Id}\|_{\phi_2(\bar{B}) \cup \phi_2 \circ F(i)(K^{i+1})} < \mu_{i+1}/2 \quad \text{and} \quad \phi_3 \circ \phi_2(T^{i+1}) \in \phi_1^{-1}(B). \tag{6.6}$$

Claim. $F_{i+1} = \phi_1 \circ \phi_3 \circ \phi_2$ satisfies the induction hypothesis for $i+1$.

Note that by choice F_{i+1} satisfy properties (ii)–(v), (vii) and (viii). Let $z \in \bar{B}$

$$\|\phi_3 \circ \phi_2(z) - z\| \le \|\phi_3 \circ \phi_2(z) - \phi_2(z)\| + \|\phi_2(z) - z\| < \mu_{i+1}.$$

Also $\phi_3 \circ \phi_2(\bar{B}) \subset B_{\delta_{i+1}}$ and from the choice of μ_{i+1} it follows that

$$\|S_{i+1} \circ \phi_3 \circ \phi_2(z) - S_{i+1}(z)\| \le \delta_{i+1}/2.$$

Thus

$$\|\phi_1 \circ \phi_3 \circ \phi_2(z) - S_{i+1} \circ \phi_3 \circ \phi_2(z)\| < \delta_{i+1}/2.$$

Hence $\|F_{i+1} - S_{i+1}\|_{\bar{B}} < \delta_{i+1}$, i.e., (i) is true. Also from relation (3.2), $\bar{B} \subset F_{i+1}^{-1}(\bar{B})$, i.e., $F(i)^{-1}(\bar{B}) \subset F(i+1)^{-1}(\bar{B})$. Thus for any $z \in B^k(0; i+1)\backslash F(i+1)^{-1}(\bar{B})$ means either $z \in int(K^i) \subset K^{i+1}$ or $\text{dist}(z, F(i)^{-1}(P^{i+1})) < \epsilon_{i+1}$. But $F(i)^{-1}(P^{i+1}) \in K^{i+1}$, hence (vi) is satisfied. Finally as $F(i)(K^{i+1}) \subset B^{i+1}$, it follows from (6.4)–(6.6), $F_{i+1}(F(i)(K^{i+1})) \subset B^k(q_{i+1}, R_{i+1})$, which proves (ix).

Hence we obtain a sequence $\{F_n\} \subset \mathsf{Aut}_0(\mathbb{C}^k)$ such that:

- From property (i) and Theorem **1.2**, $\Omega_{\{F_n\}} \cong \Omega_{\{S_n\}}$,
- From property (iii) and (v), $K \subset \partial\Omega_{\{F_n\}}$ where $K = \cup_{i=0}^{\infty} \partial K^i$,
- From property (vi), K is a dense subset of $\partial\Omega_{\{F_n\}}$.

Now by construction, the $2k$-dimensional measure is non–zero at every point of K and hence on $\partial\Omega_{\{F_n\}}$. Thus the proof. □

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