MOSCOW MATHEMATICAL JOURNAL
Volume 15, Number 2, April–June 2015, Pages 187–203

STABILITY CONDITIONS FOR SLODOWY SLICES AND REAL VARIATIONS OF STABILITY

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To Borya Feigin with gratitude and best wishes on his anniversary

Abstract. The paper provides new examples of an explicit submanifold in Bridgeland stabilities space of a local Calabi–Yau.

More precisely, let X be the standard resolution of a transversal slice to an adjoint nilpotent orbit of a simple Lie algebra over $\mathbb{C}$. An action of the affine braid group on the derived category $D^b(Coh(X))$ and a collection of t-structures on this category permuted by the action have been constructed earlier by the last two authors and S. Riche. In this note we show that the t-structures come from points in a certain connected submanifold in the space of Bridgeland stability conditions. The submanifold is a covering of a submanifold in the dual space to the Grothendieck group, and the affine braid group acts by deck transformations.

We also propose a new variant of definition of stabilities on a triangulated category, which we call a "real variation of stability conditions" and discuss its relation to Bridgeland's definition. The main theorem provides an illustration of such a relation. We state a conjecture by the second author and A. Okounkov on examples of this structure arising from symplectic resolutions of singularities and its relation to equivariant quantum cohomology. We verify this conjecture in our examples.

2010 Math. Subj. Class. 14F05, 14N35, 17B08, 17B50.

Key words and phrases. Stability conditions on triangulated categories, Slodowy slices, symplectic resolutions, (affine) braid group actions, quantization in positive characteristic, quantum cohomology.

1. Introduction

The goal of this work is twofold. One aim, achieved in Theorem 1 is to describe new examples of an explicit connected submanifold in the space of locally finite Bridgeland stabilities on the derived category of a local Calabi–Yau manifold X. The examples in question come from a simple algebraic group G, more precisely X is the resolution of a transversal slice to an adjoint orbit in the nilpotent cone of a simple algebraic group. It is well know that minimal resolutions of Kleinian surface

Received April 9, 2014.
The second and third named authors were supported by NSF grants.

singularities belong to this class, in that special case we recover a weaker variant of Bridgeland's result [Br5], see Example 1. Notice that the real dimension of the submanifold in the space of stabilities we describe is twice the second Betti number of the flag variety of G; in almost all cases (see footnote to Theorem 2 for more precise information) this is equal to twice the second Betti number of X. It is our understanding that mirror symmetry conjectures suggest existence of a canonical submanifold in the stability space $\mathrm{Stab}(D^b(Coh(M)))$ for a Calabi–Yau manifold M, called the stringy moduli space, whose real dimension is $2b_2(M)$: under mirror duality it should correspond to a covering of the moduli space of deformations of the dual Calabi–Yau manifold. It seems natural to expect that the submanifold we describe is related to the stringy moduli space of X.

The second goal is to explore the parallelisms between structures discovered, respectively, by representation theorists working in Kazhdan–Lusztig theory and its generalizations and by algebraic geometers studying Bridgeland stability conditions. The first point to mention here is the action of the braid group (or its generalizations) on the derived category of modules, which is the key ingredient in Kazhdan–Lusztig theory. The braid group can be described as the fundamental group of a space which, in our opinion, should be thought of as the counterpart of the space $\mathcal{K}$ of Kaehler parameters in algebraic geometry, the action of the fundamental group $\pi_1(\mathcal{K})$ on the derived category of coherent sheaves has been constructed in a number of examples (see references in Section 3.2). Localization theory in positive characteristic [BMR1] relates derived category of modules to that of coherent sheaves, in this setting the two constructions can be directly related as explained below.

Furthermore, as the proof of the Theorem 1 shows, in some examples a variant of the *central charge* map appearing in Bridgeland's definition arises naturally from polynomials describing *dimensions of modules* in positive characteristic. These examples led us to introduce new definitions of structures related to but different from Bridgeland stability manifold: these are (symmetric) real variations of stability conditions and local systems of categories with stabilities.

We discuss heuristic relation of our definitions to Bridgeland's one in Remark 4, while comparing Theorem 1 to Theorem 2 one gets an example of a precise connection between the two. We also state a conjecture (Conjecture 1 in section 3.3) due to A. Okounkov and the second author on existence of such a structure on derived categories of general symplectic resolutions and their relation to equivariant quantum cohomology. We explain in Theorem 2 that validity of this conjecture in our examples follows from known results.

Acknowledgements. We thank Andrei Okounkov for the permission to quote a conjecture partly due to him and Tom Bridgeland for help with the proof of Lemma 2. The second author thanks Leonid Rybnikov for discussions leading to conjecture in Remark 6. A part of the work on the paper was done while the last two authors enjoyed the hospitality of Institute for Advanced Study at Hebrew University of Jerusalem, they would like to thank that Institution for excellent work conditions.

We are grateful for the opportunity to dedicate this work to Borya Feigin; his unique vision of things in and beyond mathematics has been an important influence

and a source of inspiration for the authors at various stages of their professional life.

2. Bridgeland Stabilities for the Resolutions of Transversal Slices

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, let $e \in \mathfrak{g}$ be a non-principal nilpotent element, $Y \subset \mathfrak{g}$ be a transversal (Slodowy) slice to the G-orbit of e. Let $\mathcal{B} = G/B$ be the flag variety of G, and $\pi\colon T^*(\mathcal{B}) \to \mathfrak{g}$ be the Springer (moment) map. Set $\mathcal{B}_e = \pi^{-1}(e)$ and $X = \pi^{-1}(Y)$. Set $\mathcal{C} = D^b(Coh_{\mathcal{B}_e}(X))$ where $Coh_{\mathcal{B}_e}(X)$ is the category of coherent sheaves on X supported on $\mathcal{B}_e$.

Certain t-structures on $\mathcal{C}$ were constructed in [BM] (announced in [B]). In this note we show that they arise from an explicit connected subset of the space $\mathrm{Stab}(\mathcal{C})$ of locally finite Bridgeland stability conditions on $\mathcal{C}$. To state the result we need more notations.

Let $\mathfrak{h}$ denote the (abstract) Cartan algebra of $\mathfrak{g}$.

We have $\mathfrak{h}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ where Λ is the weight lattice. Let $\mathfrak{h}^*_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathfrak{h}^*$ be the real dual Cartan. The affine Weyl group W_{aff} acts on $\mathfrak{h}^*$ and on $\mathfrak{h}^*_{\mathbb{R}}$ by affine-linear transformations. Let $\mathfrak{h}^*_{\mathrm{reg}}$ be the union of free orbits of W_{aff} on $\mathfrak{h}^*$, thus $\mathfrak{h}^*_{\mathrm{reg}}$ is the complement to the affine coroot hyperplanes $H_{\check{\alpha},n} = \{\lambda \in \mathfrak{h}^*\colon \langle \lambda, \check{\alpha}\rangle = n\}$, where $n \in \mathbb{Z}$ and $\check{\alpha}$ is a coroot.

2.1. Action of the affine braid group and t-structures assigned to alcoves. Recall that an *alcove* is a connected component of $\mathfrak{h}^*_{\mathbb{R},\mathrm{reg}} = \mathfrak{h}^*_{\mathrm{reg}} \cap \mathfrak{h}^*_{\mathbb{R}}$. The natural action of the affine Weyl group on $\mathfrak{h}^*$ induces a simply transitive action of W_{aff} on the set of alcoves. We denote this set by Alc.

The argument below is based on the construction of [BM] which assigns a t-structure τ_A on $\mathcal{C}$ to an alcove $A \in \mathrm{Alc}$. The t-structure τ_A can be described using derived localization over a field of characteristic $p > 0$ [BMR1]. Roughly speaking, modules for the sheaf $D_\lambda(\mathcal{B})$ of twisted differential operators on $\mathcal{B}$ are closely related to coherent sheaves on $T^*(\mathcal{B})$; on the other hand, the derived category $D^b(D_\lambda(\mathcal{B}))$ is identified with the derived category of an appropriate quotient of the enveloping algebra $U\mathfrak{g}$. Thus one can get a t-structure on $D^b(Coh(T^*(\mathcal{B})))$ which is compatible with the tautological t-structure on $D^b(U\mathfrak{g}\text{-mod})$. The t-structure τ_A arises this way when the twisting parameter λ satisfies the condition $\frac{\lambda+\rho}{p} \in A$. There exists also a more direct construction of the t-structure τ_A over a characteristic zero field, though available proof of its properties relies on positive characteristic picture.

For future reference we mention that τ_A is compatible with a t-structure on $D^b(Coh(X))$ which corresponds to the tautological t-structure under an equivalence $D^b(Coh(X)) \cong D^b(R\text{-mod}_{fg})$ where R is a certain $\mathcal{Z}(Y)$-ring which is finite as an $\mathcal{Z}(Y)$ module and $R\text{-mod}_{fg}$ denotes the category of finitely generated R-modules. It follows that the heart of τ_A is a finite length abelian category.

Let $B_{\mathrm{aff}} = \pi_1(\mathfrak{h}^*_{\mathrm{reg}}/W_{\mathrm{aff}})$ be the affine braid group (this is the affine braid group of Langlands dual group in the standard terminology). An action of B_{aff} on $\mathcal{C}$ was defined in [BR]. This action permutes the t-structures τ_A. More precisely, to each pair of alcoves A, $A' \in \mathrm{Alc}$ one can assign an element $b_{A,A'} \in B_{\mathrm{aff}}$; it is then shown in [BM] that $b_{A,A'}$ sends τ_A to $\tau_{A'}$. To define $b_{A,A'}$ notice that an element in B_{aff}

is determined by a homotopy class of a path connecting two alcoves in $\mathfrak{h}^*_{\mathrm{reg}}$. The element $b_{A,A'}$ corresponds to a path $\phi\colon [0,1] \to \mathfrak{h}^*_{\mathrm{reg}}$ such that $\phi(0) \in A$, $\phi(1) \in A'$ and $\phi(t) \in \mathfrak{h}^*_{\mathbb{R}} + i(\mathfrak{h}^*_{\mathbb{R}})^+$ for $t \in (0,1)$; here $(\mathfrak{h}^*_{\mathbb{R}})^+ \subset \mathfrak{h}^*_{\mathbb{R}}$ is the dominant Weyl chamber. This requirement characterizes the homotopy class of ϕ uniquely.

For future reference we fix a universal covering $\widetilde{\mathfrak{h}^*_{\mathrm{reg}}}$. We also fix a continuous lifting of each alcove $A \in \mathrm{Alc}$ to a subset $\widetilde{A}$ in $\widetilde{\mathfrak{h}^*_{\mathrm{reg}}}$, so that for each two alcoves A, A' a path representing $b_{A,A'}$ lifts to a continuous path connecting $\widetilde{A}$ to $\widetilde{A}'$.

2.2. Embedding $\mathfrak{h}^* \to K^0(\mathcal{C})^*$ and the "quasi-exponential" map. We identify $H^*(G/B,\mathbb{C})$ with $K^0(Coh(G/B)) \otimes \mathbb{C}$ by means of the Chern character map. Notice that the class of the line bundle $\mathcal{O}(\lambda)$ attached to $\lambda \in \Lambda$ corresponds to $\exp(\lambda) \in H^*(G/B)$ where $\lambda \in \Lambda$ is considered as an element in $\mathfrak{h}^* = H^2(G/B)$; it is a nilpotent element in the commutative algebra $H^*(G/B)$, so its exponent is well defined.

The formula $([\mathcal{F}],[\mathcal{G}]) = \chi(\mathrm{pr}^*(\mathcal{F}) \otimes \mathcal{G})$ defines a bilinear pairing $K^0(G/B) \times K^0(\mathcal{C}) \to \mathbb{Z}$. Here χ stands for Euler characteristic and pr for the projection $T^*(G/B) \to G/B$. This gives a map $H^*(G/B) \to (K^0(\mathcal{C}) \otimes \mathbb{C})^*$. We will omit complexification from notation where it is not likely to lead to a confusion, and identify an element in $H^*(G/B)$ with its image in $K^0(\mathcal{C})^*$.

We extend the map $\Lambda \to H^*(G/B) \cong K^0(Coh(G/B)) \otimes \mathbb{C}$, $\lambda \mapsto \exp(\lambda)$ to $\mathfrak{h}^*$ as follows. Define the "quasi-exponential" map $E\colon \mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{R}} \times (\sqrt{-1}\,\mathfrak{h}^*_{\mathbb{R}}) \to H^*(G/B)$ by:

$$E\colon x + \sqrt{-1}\,y \mapsto \exp(x)(1 + \sqrt{-1}\exp(y)).$$

In fact, the map

$$x + \sqrt{-1}\,y \mapsto (x,\, x+y) \tag{1}$$

is a W_{aff} equivariant isomorphism between $\mathfrak{h}^*$ and $(\mathfrak{h}^*_{\mathbb{R}})^2$, where W_{aff} acts on $(\mathfrak{h}^*_{\mathbb{R}})^2$ diagonally. Written as a map from $(\mathfrak{h}^*_{\mathbb{R}})^2$, the map E takes the form $(\lambda,\mu) \mapsto \exp(\lambda) + \sqrt{-1}(\exp(\mu))$.

Remark 1. A variation of the argument below also works for the map $E(z) = \exp(z)$ (with a less explicit and not necessarily open, though still connected neighborhood V of $(\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}}$). The proof of the statement involving the above quasi-exponential map is a bit shorter, so we opted for presenting that version of the result.

Lemma 1. *The map E is compatible with the W_{aff} action where the action on the source is the standard affine linear action on $\mathfrak{h}^*$, and the one on the target is induced by the B_{aff} action on $D^b(Coh(T^*(\mathcal{B}))$ from* [BM].

Proof. Translations act on the target by twisting with a line bundle and on the source by shifting the real part, thus it is easy to deduce that the map is compatible with the action of the lattice of translations. Compatibility with the action of the finite Weyl group W follows from [BM, Theorem 1.3.2]. □

2.3. The main result. Define the "almost regular" part $(\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}}$ of the real Cartan as the set of points in $\mathfrak{h}^*_{\mathbb{R}}$ whose stabilizer in W_{aff} has at most two elements. For

$\lambda,\, \mu \in \mathfrak{h}^*_{\mathbb{R}}$ we will write $\lambda \preceq \mu$ if λ lies in the closure of the face which contains μ. Here by a face we mean a stratum of the stratification of $\mathfrak{h}^*_{\mathbb{R}}$ cut out by the coroot hyperplanes (thus alcoves are faces of maximal dimension). Define a neighborhood V of the set of almost regular real points $(\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}}$ in $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ by:

$$V = \{(\lambda,\, \mu) \in (\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}} \times \mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*_{\mathbb{R}} \times \mathfrak{h}^*_{\mathbb{R}} \cong \mathfrak{h}^* \colon \lambda \preceq \mu \vee \lambda \in (\mathfrak{h}^*_{\mathbb{R}})^{reg}\},$$

where we used the isomorphism inverse to (1).

Thus V is an open neighborhood of $(\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}}$ in $\mathfrak{h}^*$. Let $V^{\mathrm{reg}} = V \cap \mathfrak{h}^*_{\mathrm{reg}}$; we have:

$$V^{\mathrm{reg}} = \{(\lambda,\, \mu) \in (\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}} \times \mathfrak{h}^*_{\mathbb{R}} \colon (\lambda \in (\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{reg}}) \vee (\lambda \in \bar{A},\, \mu \in A \text{ for some } A \in \mathrm{Alc})\}.$$

Let $\widetilde{V^{\mathrm{reg}}}$ be the preimage of V^{reg} in $\widetilde{\mathfrak{h}^*_{\mathrm{reg}}}$.

Theorem 1. *There exists a unique map ι from $\widetilde{V^{\mathrm{reg}}}$ to the space* $\mathrm{Stab}(\mathcal{C})$ *of locally finite Bridgeland stability conditions on $\mathcal{C}$ such that*:

1. *The composed map $Z \circ \iota$, where Z is the projection* $\mathrm{Stab} \to K^0(\mathcal{C})^*$, *coincides with the map $\sqrt{-1}E \circ \pi$ where π is the projection $\widetilde{\mathfrak{h}^*_{\mathrm{reg}}} \to \mathfrak{h}^*_{\mathrm{reg}}$.*
2. *For some (equivalently, for any) $A \in \mathrm{Alc}$ and $z \in \widetilde{A}$ the underlying t-structure of the stability $\iota(z)$ coincides with τ_A.*

The map ι is compatible with the action of B_{aff} which acts on the source by deck transformations, while the action on the target comes from the action on the category $\mathcal{C}$.

Proof. Uniqueness of a map ι satisfying (1) and (2) for some fixed alcove A and $z \in \widetilde{A}$ follows from [Br2, Theorem 1.2] which asserts that the map Z is a local homeomorphism. It remains to show existence of a B_{aff} equivariant map ι which satisfies (1) and (2) for all $A,\, z \in \widetilde{A}$. This will be done in section 4.2.

Example 1. If $\dim(X) = 2$, i.e., e is sub-regular, X is well known to be the minimal resolution of a Kleinian singularity. In this case a component of the space $\mathrm{Stab}(\mathcal{C})$ was described in [Br5] (in the special case of type A it is shown in [IUU] that this component equals the whole space $\mathrm{Stab}(\mathcal{C})$). It is easy to see that our submanifold is contained in the one described in *loc. cit.*

Remark 2. It is easy to show that $\pi_1(V^{\mathrm{reg}}/W_{\mathrm{aff}})$ is a free group with $rank(G)$ generators. This group surjects to $B_{\mathrm{aff}} = \pi_1(\mathfrak{h}^*_{\mathrm{reg}}/W_{\mathrm{aff}})$. (The same remains true if V is replaced by any sufficiently small convex W_{aff} invariant neighborhood of $(\mathfrak{h}^*_{\mathbb{R}})^{\mathrm{ar}}$ in $\mathfrak{h}^*$). Thus the covering $\widetilde{V^{\mathrm{reg}}} \to V^{\mathrm{reg}}$ is connected but it is far from being universal. It would be interesting to construct an explicit subset in $\mathrm{Stab}(\mathcal{C})$ which is a universal covering of a domain whose fundamental group is isomorphic to the affine braid group.

Remark 3. By a standard argument (cf. [Br5], [Br3]) injectivity of the map ι is equivalent to the fact that the orbit of τ_A under the action of B_{aff} on the set of t-structures on $\mathcal{C}$ is free. Notice that the result of [ST] in type A and of [BT] in general implies (at least if G is simply-laced) that the orbit of τ_A under the action of the subgroup $B \subset B_{\mathrm{aff}}$ is free; here B is the Artin braid group associated to the finite Dynkin graph.

3. Real Variations of Stabilities

In this section we discuss the motivation for the main result and suggest some new definitions. We also describe a conjecture by A. Okounkov and the second author and verify it in our examples.

3.1. Real variations of stability conditions: definition. We expect that the following structure is relevant, at least in some examples, for understanding the aspects of Calabi–Yau categories which have been studied in the literature via the concept of Bridgeland stabilities.

Let $\mathcal{C}$ be a finite type triangulated category and V a real vector space. Suppose that a discrete collection Σ of affine hyperplanes in V is fixed, let V^0 denote their complement. For each hyperplane in Σ consider the parallel hyperplane passing through zero, let Σ_{lin} be the set of those linear hyperplanes. Fix a component V^+ of the complement to the union of hyperplanes in Σ_{lin}. The choice of V^+ determines for each $H \in \Sigma$ the choice of the positive half-space $(V \setminus H)^+ \subset V \setminus H$, where $(V \setminus H)^+ = H + V^+$. By an *alcove* we mean a connected component of the complement to hyperplanes in Σ and we let Alc denote the set of alcoves. For two alcoves A, $A' \in$ Alc sharing a codimension one face which is contained in a hyperplane $H \in \Sigma$ we will say that A' is *above* A and A is below A' if $A' \in (V \setminus H)^+$.

Definition 1. A *real variation of stability conditions* on $\mathcal{C}$ parametrized by V^0 and directed to V^+ is the data (Z, τ), where Z (the central charge) is a polynomial map $Z\colon V \to (K^0(\mathcal{C}) \otimes \mathbb{R})^*$, and τ is a map from Alc to the set of bounded t-structures on $\mathcal{C}$, subject to the following conditions.

1. If M is a nonzero object in the heart of $\tau(A)$, $A \in$ Alc, then $\langle Z(x), [M]\rangle > 0$ for $x \in A$.
2. Suppose A, $A' \in$ Alc share a codimension one face H and A' is above A. Let $\mathcal{A}$ be the heart of $\tau(A)$; for $n \in \mathbb{N}$ let $\mathcal{A}_n \subset \mathcal{A}$ be the full subcategory in $\mathcal{A}$ given by: $M \in \mathcal{A}_n$ if the polynomial function on V, $x \mapsto \langle Z(x), [M]\rangle$ has zero of order at least n on H. One can check that $\mathcal{A}_n$ is a Serre subcategory in $\mathcal{A}$, thus $\mathcal{C}_n = \{C \in \mathcal{C}\colon H^i_{\tau(A)}(C) \in \mathcal{A}_n\}$ is a thick subcategory in $\mathcal{C}$. We require that
   (a) The t-structure $\tau(A')$ is compatible with the filtration by thick subcategories $\mathcal{C}_n$.
   (b) The functor of shift by n sends the t-structure on $\mathrm{gr}_n(\mathcal{C}) = \mathcal{C}_n/\mathcal{C}_{n+1}$ induced by $\tau(A)$ to that induced by $\tau(A')$. In other words,
   $$\mathrm{gr}_n(\mathcal{A}') = \mathrm{gr}_n(\mathcal{A})[n]$$
   where $\mathcal{A}'$ is the heart of $\tau(A')$, $\mathrm{gr}_n = \mathcal{A}'_n/\mathcal{A}'_{n+1}$, $\mathcal{A}'_n = \mathcal{A}' \cap \mathcal{C}_n$.

Remark 4. Requirement (1) of the definition implies that $(\sqrt{-1}\,Z, \tau)$ define a map from V^0 to the space of Bridgeland stabilities $\mathrm{Stab}(\mathcal{C})$. Since V^0 is disconnected, this structure by itself does not provide any relation between the different t-structures, thus it is too weak to yield interesting results. Axiom (2) connects the t-structures assigned to different connected components of V^0; it is based on the same intuition as Bridgeland's definition (as we understand it): as x travels from A to A' in the complexification $V_{\mathbb{C}} \setminus H_{\mathbb{C}}$ the phase of a stable objects in $\mathcal{C}_n \setminus \mathcal{C}_{n+1}$ is shifted by $n\pi$, hence the homological shift by n in requirement (2). This heuristics suggests that given a real variation of stability conditions one might expect a map from a connected covering of the complexification $V_{\mathbb{C}}^0 = V_{\mathbb{C}} \setminus \bigcup_{H\in\Sigma} H_{\mathbb{C}}$ to $\mathrm{Stab}(\mathcal{C})$ sending a point x in an alcove A to the stability corresponding to the central charge $Z(x)$ and t-structure τ_A (notice that the choice of V_+ defines a homotopy class of a path in $V_{\mathbb{C}}^0$ between any two alcoves, it is fixed by the requirement that the path maps $(0,\,1)$ to $V+iV_+$; this defines compatible lifting of all alcoves to the covering). The main Theorem of this note is a partial result in that direction. However, the fact that we get a map from a covering of a proper subset in $V_{\mathbb{C}}^0$ which is not even homotopy equivalent to the whole space, and have to use a somewhat unnatural quasi-exponential map is an indication of technical difficulties in connecting the two definitions. We expect even more serious difficulties in the cases when filtrations $(\mathcal{C}_n)$ do not reduce to a two step filtration.

Instead of trying to establish a direct relation between the two structures, it may be more fruitful to view them as different implementations of the same intuition of "physical" origin and possibly try to find a common generalization of the two (cf. the end of the second paragraph in [Br4]).

Remark 5. In many cases (including the examples considered in this paper) one has natural equivalences $D^b(\mathcal{A}') \cong \mathcal{C} \cong D^b(\mathcal{A})$. The resulting equivalence $D^b(\mathcal{A}') \cong D^b(\mathcal{A})$ belongs to a class of equivalences which appeared in the work of Chuang–Rouquier and Craven–Rouquier under the name of *perverse equivalences* see [ChR], [CR] and references therein (our setting may be slightly more general, but the generalization is straightforward).

Example 2. Let $\mathcal{C} = D^b(Coh_{\mathcal{B}_e}(X))$ as above, $V = \mathfrak{h}_{\mathbb{R}}^*$ and let Σ consist of the affine coroot hyperplanes. Let V^+ be the positive Weyl chamber. Let $\tau\colon A \to \tau_A$ be the map described in [BM, 1.8]. The polynomial map $Z\colon \mathfrak{h}_{\mathbb{R}}^* \to K^0(\mathcal{C})_{\mathbb{R}}^*$ is characterized uniquely by its values at the points of the lattice $\Lambda \subset \mathfrak{h}^*$; these values are given by

$$\langle Z(\lambda),\, [\mathcal{F}]\rangle = \chi(\mathcal{F}\otimes\mathcal{O}(\lambda)), \tag{2}$$

where χ denotes the Euler characteristic and $\mathcal{O}(\lambda)$ is the line bundle attached to λ.

Proposition 1 below implies that this data provides an example of a real variation of stability conditions, see Theorem 2 for a stronger statement. Notice that in this case for every pair of neighboring alcoves as in part 2 of the definition the filtration $\mathcal{C}_n$ is just a two term filtration, i.e., $\mathcal{C}_2 = \{0\}$. Another special feature of this example is that all the t-structures $\tau(A)$ lie in one orbit of the group of automorphisms of $\mathcal{C}$ (in fact, of the group B_{aff} acting on $\mathcal{C}$).

Remark 6. We expect Example 2 of a real variation of stability conditions to be a part of a richer structure. Namely, let $F \subset V$ be a face of an arbitrary codimension and let $A,\, A'$ be two alcoves containing F in its closure. Assume that A and A' are opposite with respect to F, i.e., that there exists a line segment $[a,\, a']$ passing through F with endpoints satisfying $a \in A,\ a' \in A'$. Let p be a path in the complexification of $[a,\, a']$ going around $\{x\} = [a,\, a'] \cap F$ along a small loop in the positive direction. We conjecture that ϕ_p is a perverse equivalence governed by the order of vanishing at x of the central charge polynomials restricted to $[a, a']$. This can likely be restated as a real variation of stabilities parametrized by the real points of a variety which maps birationally to V so that the preimage of Σ is a divisor with normal crossings. The variety is closely related to the De Concini Procesi partial compactification of $V^0_{\mathbb{C}}$ [DCP].

Remark 7. Given a real variation of stability conditions one can produce a new one by adding an arbitrary hyperplane to the collection; the new set of alcoves Alc' maps to the old one Alc, so one can define the map from Alc' to the set of t-structures by composing the given map from Alc with the map $\mathrm{Alc}' \to \mathrm{Alc}$. Let us say that a real variation of stability conditions is nondegenerate if it can not be obtained this way from another one. One can check that the real variation of stability conditions in Example 2 is nondegenerate; moreover, for each two neighboring alcoves the corresponding t-structures are different.

3.2. Real variation of stabilities and automorphisms of derived categories. In some examples in the literature (see, for example, [Br5], [Br3], [Br1], [IUU]) (a component of) the space $\mathrm{Stab}(\mathcal{C})$ is realized as a covering of a domain in $K^0(\mathcal{C})^*_{\mathbb{C}}$ where the group of automorphisms of $\mathcal{C}$ acts by deck transformations. We suggest the following counterpart of this picture in the framework of real variations of stability conditions.

Definition 2. A real variation of stability conditions is *symmetric* if the following holds.

(1) For any alcoves A, A' as in part 2 of Definition 1 there exists an autoequivalence $m_{A,A'}$ of $\mathcal{C}$ preserving the subcategories $\mathcal{C}_n \subset \mathcal{C}$, so that the induced autoequivalence of $\mathcal{C}_n/\mathcal{C}_{n+1}$ is isomorphic to the shift functor $M \mapsto M[2n]$.
(2) The autoequivalences $m_{A,A'}$ can be chosen so that the following holds. Consider the groupoid $P(V^0_{\mathbb{C}})$ whose objects are alcoves and morphisms from A to A' are homotopy classes of paths in $V^0_{\mathbb{C}}$ starting at A and ending at A'. Then there exists a functor from $F\colon P(V^0_{\mathbb{C}}) \to Cat$ where Cat is the category of categories with morphisms being the isomorphism classes of functors,[1] such that:
    (a) $F(A) = \mathcal{C}$ for all $A \in \mathrm{Alc}$.
    (b) For A, $A' \in \mathrm{Alc}$ as above F sends the class of a path going from A to A' around H in the positive direction to the identity functor.
    (c) For A, A' as above F sends the class of a path going from A' to A in the positive direction to $m_{A,A'}$.

Remark 8. Groupoid $P(V^0_{\mathbb{C}})$ appearing above admits an explicit description in terms of generators and relations generalizing a standard presentation of the (affine) braid group, see [Sa].

[1]For the sake of brevity here and in Definition 3 we present a weak version of the definition which only deals with isomorphism classes of equivalences, we do not address the issue of fixing the isomorphisms between the equivalences in a compatible way. One can upgrade it to a definition of a finer structure involving the 2-category of categories, or a version of the formalism of infinity categories.

Data as in Definition 2 yields the following more symmetric structure which does not involve the choice of the "positive cone" V^+.

Definition 3. Let V, Σ be as in Definition 1. A *local system of categories with a stability condition* is the following collection of data.

To an alcove A one assigns a triangulated category $\mathcal{C}_A$ with a bounded t-structure τ_A whose heart is denoted by $\mathcal{A}_A$. To every homotopy class of a path p connecting A to A' in $V^0_{\mathbb{C}}$ there corresponds a triangulated equivalence $\phi_p \colon \mathcal{C}_A \to \mathcal{C}_{A'}$, which combine into a functor $P(V^0_{\mathbb{C}}) \to Cat$ (notations of Definition 2), i.e., we have $\phi_{pp'} \cong \phi_p \circ \phi_{p'}$ and $\phi_p \cong \mathrm{Id}_{\mathcal{C}_A}$ for a loop $p\colon [0, 1] \to A$.

We assume that for any path p connecting an alcove A to itself the induced automorphism of $K^0(\phi_p)\colon K^0(\mathcal{C}_A) \to K^0(\mathcal{C}_A)$ equals identity. We set $K = K^0(\mathcal{C}_A)$ for some alcove A, thus $K \cong K^0(\mathcal{C}_A)$ canonically for any alcove A.

Finally, we assume that a polynomial *central charge* map $Z\colon V \to (K \otimes \mathbb{R})^*$ is fixed so that

(1) $\langle Z(x), [M]\rangle > 0$ for $x \in A$, $M \in \mathcal{A}_A$, $M \neq 0$.
(2) Let A and A' be two alcoves sharing a codimension one face H. Let p be the path from A to A' going around H in the positive direction. Then the t-structure τ_A on $\mathcal{C}_A$ is related to the image of the t-structure $\tau_{A'}$ under ϕ_p^{-1} as spelled out in Definition 1(2).

As was pointed out above, a symmetric real variation of stabilities yields a local system of categories with a stability condition by setting $\mathcal{C}_A := \mathcal{C}$, $\tau_A := \tau(A)$ etc.

Conversely, it is easy to see that given a local system of categories with a stability condition together with an additional choice of a cone V^+ as in Definition 1 one can obtain a symmetric real variation of stabilities in the following way. For any two alcoves A, A' there exists a unique homotopy class of a path $p_{A,A'}\colon [0, 1] \to V^0_{\mathbb{C}}$ such that $p\colon (0, 1) \to V + \sqrt{-1}\, V_+$; moreover, we have $p_{A,A''} \sim p_{A',A''} p_{A,A'}$ for any triple of alcoves A, A', A''. Thus we can fix a triangulated category $\mathcal{C}$ together with equivalences $\phi_A\colon \mathcal{C} \to D^b(\mathcal{A}_A)$, so that $\phi_{A'}\phi_A^{-1} \cong \phi_{p_{A,A'}}$. The category $\mathcal{C}$ then comes equipped with the data described in the definition of a symmetric real variation of stabilities.

3.3. Symmetric real variation of stabilities and symplectic resolutions. Let $\pi\colon X \to Y$ be a conical (homogeneous) symplectic resolution over the field k. Recall that this means that π is a resolution of singularities, X carries an algebraic symplectic form ω and the multiplicative group $\mathbb{G}_m$ acts on X, Y compatibly, dilating the symplectic form so that $t^*(w) = t^2\omega$. The action of $\mathbb{G}_m$ on Y is assumed to contract Y to a point $0 \in Y$; this implies that Y is affine. We set $\mathcal{C} = D^b(Coh_{\pi^{-1}(0)}(X))$.

Along with $\mathcal{C}$ we will also need its graded version $\mathcal{C}_{\mathrm{gr}} := D^b(Coh^{\mathbb{G}_m}_{\pi^{-1}(0)}(X))$. Notice that $K^0(\mathcal{C}_{\mathrm{gr}})$ is module over $\mathbb{Z}[q, q^{-1}]$ where q acts by twisting with the tautological character of $\mathbb{G}_m$. For $t \in \mathbb{C}^\times$ we set $K(\mathcal{C}_{\mathrm{gr}})_t = K(\mathcal{C}_{\mathrm{gr}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$, where $\mathbb{Z}[q, q^{-1}]$ maps to $\mathbb{C}$ by sending q to t.

Assume for simplicity that $\mathrm{Pic}(X) \cong \mathbb{Z}^r$. Set $V = \mathrm{Pic}(X) \otimes \mathbb{R}$; let V^+ is spanned by the ample cone, and let central charge map Z be given by formula (2).

We now state a conjecture due to the second author and A. Okounkov, see also discussion in [BMO, Section 1.10], [MO, Section 1.1.6] etc.

In the statement we need some basic information about quantizations in positive characteristic. It is known that for k of characteristic zero, hence also of large positive characteristic, $H^i(X, \mathcal{Z}) = 0$ for $i > 0$, thus X is admissible in the sense of [BK2, Definition 1.21]. Thus the main result of [BK2] implies existence of a canonical quantization of X over k of characteristic $p > 0$. Furthermore, [BK2, Theorem 1.23] and discussion following its proof show that for $\lambda \in \mathrm{Pic}(X)$ one can twist the canonical quantization by λ, we call the result the quantization with parameter λ. All these quantizations are Frobenius constant (see [BK1], [BK2] for the definition) thus the result is an Azumaya algebra on the Frobenius twist $X^{(1)}$ of X. We denote this Azumaya algebra by $\mathcal{A}_\lambda$, we also fix an isomorphism of schemes $X \cong X^{(1)}$. Notice that the isomorphism class of $\mathcal{A}_\lambda$ depends only on the image of λ in $\mathrm{Pic}(X)/p\,\mathrm{Pic}(X)$; however, we view λ, not that image, as a parameter for quantization, this allows us to fix a compatible system of Morita equivalences $A_\lambda \sim A_\mu$, λ, $\mu \in \mathrm{Pic}(X)$, these Morita equivalences are used in making sense of compatibility for splitting of Azumaya algebras in part (3) of the following conjecture.

Conjecture 1. *The above data of $\mathcal{C}$, V, Z, V^+ admits a natural extension to a symmetric real variation of stability conditions with the following properties.*

(1) *The action of $\pi_1(V^0_{\mathbb{C}})$ on $\mathcal{C}$ lifts to an action on $\mathcal{C}_{\mathrm{gr}}$.*
(2) *Assume that $k = \mathbb{C}$. Set $T = N(X) \otimes \mathbb{C}^*$ and set*

$$D = \bigcup_{H \in \Sigma} \{\exp(2\pi i h) \colon h \in H\}.$$

Then the (small) $\mathbb{C}^*$-equivariant quantum cohomology of X defines a family of flat connections on the trivial vector bundle over T with fiber $H^*(X)$ depending on a parameter h (the equivariant parameter). These connections have regular singularities on D.

We have $\pi_1(V^0_{\mathbb{C}}) \subset \pi_1(T \setminus D)$. For a given generic value h of the parameter the monodromy of the corresponding connection restricted to $\pi_1(V^0_{\mathbb{C}})$ is isomorphic to the induced action on $K(\mathcal{C}_{\mathrm{gr}})_{\exp(2\pi i h)}$.

(3) *Assume that k is a field of characteristic $p \gg \dim(X)$. Suppose that $\lambda \in \mathrm{Pic}(X)$ is such that $\frac{\lambda}{p} \in A$.*

The Azumaya algebra $\mathcal{A}_\lambda$ is split on the formal neighborhood of $\pi^{-1}(0)$, we fix such splitting for all $\lambda \in \mathrm{Pic}(X)$ in a compatible way. Thus we get an equivalence between $\mathcal{C}$ and the category $\mathcal{A}_\lambda\text{-mod}_{\pi^{-1}}(0)$ of coherent sheaves of $\mathcal{A}_\lambda$-modules supported on $\pi^{-1}(0)$. The composed functor

$$\mathcal{C} \cong D^b(\mathcal{A}_\lambda\text{-mod}_{\pi^{-1}(0)}) \xrightarrow{R\Gamma} D^b(\Gamma(\mathcal{A}_\lambda)\text{-mod})$$

is an equivalence of triangulated categories sending τ_A to the tautological t-structure.

(4) *Let $X_R \to Y_R$ be a conical symplectic resolution over a commutative ring R finitely generated over $\mathbb{Z}$. Then possibly after a finite localization of R, the symmetric real variations of stabilities from (2) and (3) exist for all base*

changes $X_k \to Y_k$ for an algebraically closed field k, and data for different k are compatible as follows.

The set Σ does not depend on k, and for each alcove A there exists a t-structure τ_A^R on $D^b(Coh_{\pi^{-1}(0)}(X_R))$ compatible under base change with the t-structure τ_A on $D^b(Coh_{\pi^{-1}(0)}(X_k))$.

Remark 9. The idea that stability conditions should be related to quantum cohomology goes back to Bridgeland (see, for example, [Br4, Section 7.2]) who proposed it based on heuristics of mirror symmetry. However, our context differs from that considered by Bridgeland in several aspects; in particular, it is essential for us to work with *equivariant* quantum cohomology, while there seems to be no available description of the (conjectural) mirror dual counterpart of equivariant quantum cohomology of a local Calabi–Yau; here we refer to equivariance with respect to an action of the multiplicative group dilating the volume form.

Theorem 2. *Let $X \to Y$ be as in section 2 and assume that the map $H^2(G/B) \to H^2(X)$ is an isomorphism.*[2] *Then Conjecture 1 holds.*

Proof. Existence of a symmetric real variation of stability conditions with Σ being the collection of affine coroot hyperplanes, as well as its relation to quantization in positive characteristic described in property (3), follow from Proposition 1 and its proof. The lifting of the B_{aff} action to the equivariant category is addressed in [BM], this yields part (1). The t-structures are constructed in [BM] for slices to nilpotent elements defined over $\mathbb{Z}[\frac{1}{h!}]$ where h is the Coxeter number, so part (4) is also clear. Finally, equivariant quantum cohomology of X was computed under the above assumptions in [BMO]. The monodromy of the resulting connection (called the affine KZ, or trigonometric Dunkl) connection is isomorphic to the B_{aff} module coming from the standard module for the affine Hecke algebra by a result of [Ch], see [BMO, Proposition 2.3]. This is isomorphic to $K^0(\mathcal{C}_{\mathrm{gr}})$ by [BM, Theorem 1.3.2(b)]. □

4. Proof of Theorem 1

4.1. Positivity property.

Proposition 1. *Let $A \in \mathrm{Alc}$ and let $M \neq 0$ be an object in the heart of τ_A. Let $A' \in \mathrm{Alc}$ be a neighboring alcove separated from A by a codimension one face F.*

(a) *The function $d_M : x \mapsto \langle E(x), [M]\rangle$ is a polynomial taking positive real values on $x \in A$.*
(b) *Either d_M takes positive values on F, or d_M has a zero of order one on F.*
(c) *Assume that M is irreducible. If $d_M|_F = 0$, then $b_{A,A'}(M) = M[\pm 1]$, where the $+$ (respectively, $-$) sign should be taken if A lies above (respectively, below) A'.*

[2]This condition holds quite often, in particular it always holds if (the Dynkin graph of) G is simply-laced; see [LNS, Theorem 1.3] for the list of exceptional cases.

Otherwise $b_{A,A'}(M) = \widetilde{M}$, *where* $\widetilde{M}$ *is an object in the heart of* τ_A *which fits into one of the two exact sequences*:

$$0 \to M' \to \widetilde{M} \to M \to 0,$$
$$0 \to M \to \widetilde{M} \to M' \to 0,$$

where the first (*respectively, the second*) *exact sequence applies if* A *lies above* (*respectively, below*) A'. *Furthermore,* $d_{M'}|_F = 0$ *and* M *is the only simple quotient* (*respectively, sub*) *module of* $\widetilde{M}$.

Proof. In this proof we will work over an algebraically closed field k possibly of characteristic $p > 0$. For large p conjugacy classes of nilpotent elements in $\mathfrak{g}_k$ are in a natural bijection with those in $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$, we fix $e_k \in \mathfrak{g}_k$ in the class corresponding to the class of e.

The construction of t-structures τ_A^k, $A \in \mathrm{Alc}$ on $\mathcal{C}_k = D^b(Coh_{\mathcal{B}_{e_k}}(X_k))$ (in self-explanatory notation) is carried out in [BM] for all k except those of positive characteristic $p \leqslant h$, where h is the Coxeter number of G.

Let $\mathcal{A}_A^k$ denote the heart of τ_A^k. Then for $p \gg 0$ the set of irreducible object in $\mathcal{A}_K^k$ is in a natural bijection with irreducible objects of $\mathcal{A}_A^{\mathbb{C}}$ (see [BM, Section 5.1.4]), so that the classes of corresponding objects match under the standard identification $K^0(\mathcal{B}_{e_{\mathbb{C}}}) = K^0(\mathcal{B}_{e_k})$ explained, for example, in [BMR1, Section 7].

Fix $p \gg 0$, and let $\lambda \in \Lambda$ be such that $\frac{\lambda+\rho}{p} \in A$, where ρ is the sum of fundamental weights. Then it is shown in [BMR1, Section 6] that

$$p^{\dim \mathcal{B}} d_{\mathcal{F}}\Big(\frac{\lambda+\rho}{p}\Big) = \dim(\Gamma_\lambda(\mathcal{F})), \tag{3}$$

where $\mathcal{F} \in \mathcal{C}_k$ and $\Gamma_\lambda \colon \mathcal{C}_k \to D^b(\mathcal{M}_\lambda)$ is a functor sending $\mathcal{A}_A^k$ to $\mathcal{M}_\lambda$, inducing a full embedding $\mathcal{A}_A^k \hookrightarrow \mathcal{M}_\lambda$. Here $\mathcal{M}_\lambda$ is the category of modules over the Lie algebra $\mathfrak{g}_k$ with the fixed generalized central character (λ, e) (here λ and e are, respectively, the Harish-Chandra and the Frobenius central characters, see [BMR1, Section 1] for terminology).

Thus for a large prime number p and $M \in \mathcal{A}_A^{\mathbb{C}}$, the polynomial d_M takes a positive value $p^{-\dim \mathcal{B}} \dim(\Gamma_{p\lambda-\rho}(M_k))$ at points $\lambda \in A$ such that $p\lambda \in \Lambda$; here $M_k \in \mathcal{A}_A^k$ is an object whose class in the Grothendieck group matches that of M. Since the set of such points, for varying p, is dense, we see that $d_M(\lambda) \geqslant 0$ for any $\lambda \in A$. It remains to see that the inequality is strict.

Suppose that $d_M(\lambda_0) = 0$ for some $\lambda_0 \in A$. We claim that d_M is a harmonic polynomial, i.e., $d_M(\lambda) = \frac{1}{|W|}\sum_w d_M(\lambda + w(\mu))$ for all λ, $\mu \in \mathfrak{h}^*$. This follows from the the fact that d_M has the form $d_M(\lambda) = \langle \xi, \exp(\lambda)\rangle$, where ξ is a linear functional on $\mathrm{Sym}(\mathfrak{h}^*)/(\mathrm{Sym}(\mathfrak{h}^*)_+^W) \cong H^*(G/B)$ and $\exp\colon \mathfrak{h}^* \to \mathrm{Sym}(\mathfrak{h}^*)/(\mathrm{Sym}(\mathfrak{h}^*)_+^W)$ is the exponential map. It is clear that a harmonic polynomial which vanishes at a point but takes non-negative values on some neighborhood of that point is identically zero. This proves (a).

The proof of (b) is based on "singular localization" Theorem of [BMR2]. Namely, let $\lambda \in \Lambda$ be such that $\frac{\lambda+\rho}{p}$ lies on the boundary of an alcove A. Then we still have the functor $\Gamma_\lambda \colon \mathcal{C}_k \to D^b(\mathcal{M}_\lambda)$ which sends $\mathcal{A}_A^k$ into $\mathcal{M}_\lambda$ but now it is not necessarily conservative (i.e., it may kill some non-zero objects). The equality (3) still holds.

Furthermore, the set of irreducible objects $\mathcal{A}^k_A$ killed by Γ_λ depends only on the face F containing $\frac{\lambda+\rho}{p}$; it is also independent of $p \gg 0$, where we use the above identification of the set of irreducible objects in $\mathcal{A}^k_A$ for varying k.

Assuming that p is large enough, we see that if $\Gamma_\lambda(M) = 0$ for some $\frac{\lambda+\rho}{p} \in F$, then the polynomial d_M vanishes at all points of F with sufficiently large prime denominator, hence $d_M|_F \equiv 0$.

Otherwise d_M takes positive values at all points of F with a large prime denominator, then an argument involving harmonic polynomials as in the proof of part (a) shows it takes positive values at all points of F.

It remains to see that the order of vanishing of polynomial d_M on F can not be greater than one. We claim that this is true for any harmonic polynomial: if a harmonic polynomial P has zero of order two a more on a coroot hyperplane F, we can apply a differential operator with constant coefficients to P to get a harmonic polynomial which has the form $\alpha^2 Q_0$, where α is the equation of the hyperplane and $Q_0|_F \neq 0$. If C is the Laplace operator (the W-invariant order two operator with no constant term), then $C(\alpha^2 Q_0)|_F = C(\alpha^2) Q_0|_F \neq 0$, which contradicts the fact that W-invariant differential operators with constant coefficients and no constant term annihilate harmonic polynomials.

c) Consider first the situation over k of large positive characteristic. In this case the braid group action admits a convenient description in terms of translation functors which we presently recall.

Claim 1. *Fix λ, $\mu \in \Lambda$ with $\frac{\lambda+\rho}{p} \in A$, $\frac{\mu+\rho}{p} \in \bar{A} \setminus A$, where $\bar{A}$ is the closure of A.*

(a) *B_{aff} acts on $D^b(\mathcal{M}_\lambda)$, so that Γ_λ is compatible with the braid group action.*
(b) *We have a bi-adjoint pair of exact functors called* translation functors

$$T_{\lambda\to\mu} \colon \mathcal{M}_\lambda \to \mathcal{M}_\mu, \quad T_{\mu\to\lambda} \colon \mathcal{M}_\mu \to \mathcal{M}_\lambda.$$

The functor $T_{\lambda\to\mu}$ sends an irreducible object to an irreducible one or zero, the nonzero images of nonisomorphic irreducibles are not isomorphic.
(c) *We have $\Gamma_\mu \cong T_{\lambda\to\mu} \circ \Gamma_\lambda$.*
(d) *We now assume that $\frac{\mu+\rho}{p}$ lies in the codimension one face F separating A from A'. Set $R = T_{\mu\to\lambda} T_{\lambda\to\mu}$.*

Then we have functorial exact triangles

$$\mathrm{Id} \to R \to b_{A,A'} \to \mathrm{Id}[1] \tag{4}$$

if A is above A';

$$b_{A,A'} \to R \to \mathrm{Id} \to b_{A,A'}[1] \tag{5}$$

if A is below A'; here the map to/from Id *from/to R is the adjunction arrow.*

Proof of the Claim. The B_{aff} action is introduced in [BMR2, Theorem 2.1.4, Corollary 2.1.6], compatibility with the B_{aff} action is [BM, Proposition 1.6.4], for proof see [Ri, Section 5]: this proves (a).

Translation functors and their adjointness are discussed in this setting in [BMR2, Section 2.2.1]. The properties of translation of an irreducible module to the wall are stated in [BMR2, Section 2.2.6, Remark 1], the proof is similar to that of the corresponding fact in characteristic zero, see [BeG, Section 2.5], this yields (b).

Part (c) is [BMR2, Lemma 2.2.3(a)].

It is easy to see that in notation of [BM] we have $b_{A,A'} = \tilde{s}_\alpha^{\pm 1}$ if A and A' are separated by a face of type α; here the plus sign is chosen if and only if A is above A'. Thus part (d) follows by comparing [BMR2, Lemma 2.2.3(c)] and characterization of the affine braid group action in [BMR2, Theorem 2.1.4, Corollary 2.1.6]. □

Now we are ready to prove part (c) of the Proposition. We assume that A is above A', the other case is treated similarly. Assume $L \in \mathcal{A}_A^{\mathbb{C}}$ is such that $d_L|_F = 0$. Let L_k be the corresponding irreducible in $\mathcal{A}_A^k$. We have $\Gamma_\mu(L_k) = 0$, thus, setting $M_k = \Gamma_\lambda(L_k)$ we get $R(M_k) = 0$ and (4) shows that $b_{A,A'}(M_k) \cong M_k[1]$. Since Γ_λ is fully faithful, we see that $b_{A,A'}(L_k) = L_k[1]$.

Now assume that $d_L|_F \neq 0$, the proof of part (b) of the Proposition shows that $T_{\lambda\to\mu}(M_k) \neq 0$. Since M_k is irreducible and the adjunction map $M_k \to R(M_k)$ is nonzero, the map $M_k \to R(M_k)$ in (4) is injective, thus $b_{A,A'}(M_k) \in \mathcal{M}_\lambda$ and we have a short exact sequence in $\mathcal{M}_\lambda$:

$$0 \to M_k \to R(M_k) \to b_{A,A'}(M_k) \to 0.$$

For any irreducible $N \in \mathcal{M}_\lambda$, $N \ncong M_k$ we have

$$\operatorname{Hom}(b_{A,A'}(M_k),\, N) \subset \operatorname{Hom}(R(M_k),\, N) = \operatorname{Hom}(\Gamma_{\lambda\to\mu}(M_k),\, \Gamma_{\lambda\to\mu}(N)) = 0 \quad (6)$$

in view of Claim 1(b). Thus there exists a nonzero map $f\colon b_{A,A'}(M_k) \to M_k$. Applying $\Gamma_{\lambda\to\mu}$ to f we get a nonzero, hence surjective map from $b_{A,A'}(M_k)$ to the irreducible (by Claim 1(b)) module $\Gamma_\lambda(M_k)$. Now Claim 1(c) implies that $\Gamma_{\lambda\to\mu}(f)$ is an isomorphism, hence $\Gamma_{\lambda\to\mu}(\operatorname{Ker}(f)) = 0$.

Taking into account (6) we get that M_k is the only irreducible quotient of $b_{A,A'}(M_k)$, while the kernel of the map $b_{A,A'}(M_k) \to M_k$ is killed by $T_{\lambda\to\mu}$. Since Γ_λ is fully faithful, we see that L_k is the only irreducible quotient of $b_{A,A'}(L_k)$, while the kernel K of the map $b_{A,A'}(L_k) \to L_k$ satisfies $d_K|_F = 0$. It follows that the same is true for L. □

4.2. End of proof of the Theorem. We start with an auxiliary Lemma, probably well known to the experts.

To state it we recall notations of [Br2]. Given a stability condition σ on a triangulated category $\mathcal{C}$ one has full subcategories $\mathcal{P}(\phi)$ and $\mathcal{P}(I)$ in $\mathcal{C}$ for $\phi \in \mathbb{R}$ and an interval $I \subset \mathbb{R}$ (we will also write $\mathcal{P}_\sigma$ when we need to emphasize dependence on σ). The categories $\mathcal{P}((a,\, a+1])$ are abelian (in fact, they are hearts of bounded t-structures on $\mathcal{C}$), while the categories $\mathcal{P}((a,\, b))$ for $a < b < a+1$ are quasi-abelian. The categories $\mathcal{P}(\phi)$ are also abelian [Br2, Lemma 5.2] and simple objects of these abelian categories are called stable.

Lemma 2. (a) *Given a locally finite stability condition on a triangulated category $\mathcal{C}$ and an object $M \in \mathcal{C}$ the following are equivalent.*

(1) *M is a simple object in the quasi-abelian category $\mathcal{P}((a,\, b))$ for some a, b, $a < b < a+1$.*
(2) *M is stable.*

(b) *Suppose that M is stable with respect to a locally finite stability condition σ. Then there exists an open neighborhood U of σ in* $\mathrm{Stab}(\mathcal{C})$ *such that M is stable with respect to any $\sigma' \in U$.*

Proof. (a) It is clear from definitions that (1) implies (2). To check that (2) implies (1) assume that M is stable of phase t. By the definition of local finiteness we can find a finite length quasi-abelian category $\mathcal{B} = \mathcal{P}((t-a,\, t+a))$ $(a < \frac{1}{2})$. The object M has a finite Jordan-Hoelder series in $\mathcal{B}$. Thus there are only finitely many elements in $K^0(\mathcal{C})$ which can be represented by a subobject of M in $\mathcal{B}$, since such a class is a sum of classes of some of the simple constituents. Since M is stable, the class of a subobject $N \neq M$ has phase strictly less than t. Thus there exists $s \in (t-a,\, t)$ such that the class of any such subobject has phase less than s. Then M is irreducible in $\mathcal{B}' = \mathcal{P}((s,\, t+a))$.

(b) Using (a) we find $a,\, b,\, a < b < a+1$, such that M is simple in $\mathcal{P}_\sigma((a,\, b))$. Fix $\alpha,\, \beta$, so that $a < \alpha < \phi < \beta < b$, where ϕ is the phase of M. By the definition of topology on the set of stability conditions (cf. [Br2, Lemma 6.1]), there exists an open neighborhood U of σ such that $M \in \mathcal{P}_{\sigma'}((\alpha,\, \beta)) \subset \mathcal{P}_\sigma((a,\, b))$ for $\sigma' \in U$. Thus M is a simple object in $\mathcal{P}_{\sigma'}((\alpha,\, \beta))$ for $\sigma' \in U$, applying part (a) again we see that M is stable with respect to such σ'. □

We now complete the proof of the Theorem. We construct the required map as follows. Recall that A_0 denotes the fundamental alcove. It is easy to see that the set

$$S = \{(\lambda,\, \mu) \in \mathfrak{h}^*_{\mathbb{R}} \times \mathfrak{h}^*_{\mathbb{R}} \cong \mathfrak{h}^* \colon (\lambda \in A_0) \vee (\lambda \in \bar{A}_0,\ \mu \in A_0)\}$$

is a fundamental domain for the action of W_{aff} on V^{reg}, here we used identification (1). In fact, S is the intersection of a contractible fundamental domain for the action of W_{aff} on $\mathfrak{h}^*$ with V.

Thus a point in $\widetilde{V^{\mathrm{reg}}}$ can be represented by a pair $(b,\, x)$ where $x \in S$ and b is a homotopy class of path from A_0 to some alcove $A \in \mathrm{Alc}$ (the projection to V^{reg} is then given by $(b,\, x) \mapsto \bar{b}(x)$ where $\bar{b}$ is the element of W_{aff} corresponding to b). We define the map ι by:

$$\iota\colon (b,\, x) \mapsto \mathfrak{S}(b(\tau_{A_0}),\, \sqrt{-1}\, E(\bar{b}(x))),$$

where we use the same notation for b and the corresponding element of B_{aff}. Here $\mathfrak{S}$ denotes the stability condition obtained from the given t-structure and central charge by means of [Br2, Proposition 5.3]. Recall from section 2.1 that the heart of τ_A is a finite length abelian category, it is easy to deduce that any $\sigma \in \iota(\widetilde{V^{\mathrm{reg}}})$ is locally finite in the sense of [Br2, Definition 5.7], thus the map lands in the space $\mathrm{Stab}(\mathcal{C})$ of locally finite stability conditions.

It is clear from the definition that the map ι is B_{aff} equivariant. It remains to check that ι is continuous.

We have to check continuity at the boundary of the region corresponding to a given $b \in B_{\mathrm{aff}}$. Without loss of generality we can assume $b = 1$. Let $\boldsymbol{\lambda} \in S$ be a boundary point (i.e., $\boldsymbol{\lambda}$ is not an inner point of S). Thus $\boldsymbol{\lambda} = (\lambda,\, \mu)$, $\lambda \in F$, $\mu \in A_0$, where F is a codimension one face of A_0. The face F belongs to a unique hyperplane which corresponds to an affine coroot α attached to a vertex of the affine

Dynkin diagram. It is easy to see that a small neighborhood of $(1, \boldsymbol{\lambda})$ is contained in the union of regions corresponding to 1 and $\tilde{s}_\alpha^{-1}$ (notice that $b_{A,A'} = \tilde{s}_\alpha^{-1}$ if α corresponds to a vertex of the finite Dynkin graph, in which case A' is below A; and $b_{A,A'} = \tilde{s}_\alpha$ otherwise[3]

By [Br2, Theorem 1.2] there exists a neighborhood $\tilde{U}$ of the point $(\tau_{A_0}, \boldsymbol{\lambda})$ in $\mathrm{Stab}(\mathcal{C})$ mapping isomorphically to a neighborhood U of $\sqrt{-1}\,E(\boldsymbol{\lambda})$ in $K^0(\mathcal{C})^*$. It suffices to see that for a small enough U and a point $\tilde{z} \in \tilde{U}$ mapping to $z \in s_\alpha(S) \cap U$, the t-structure underlying $\tilde{z}$ is $\tau_{A'} = \tilde{s}_\alpha^{-1}(\tau_A)$; here A' is the neighboring alcove, $A' = s_\alpha(A_0)$.

Let M be an irreducible object. It suffices to show that $\tilde{s}_\alpha^{-1}(M)$ lies in the heart of the t-structure of $\tilde{z}$. The proof is similar to the proof of [Br5, Lemma 3.5].

Consider the dichotomy of Proposition 1(c). If $d_M|_F = 0$, then according to Proposition 1(c) $\tilde{s}_\alpha^{-1}(M) = M[-1]$. Using Lemma 2(b) we see that M is stable for $\tilde{z}$ (if $\tilde{U}$ is small enough). We have $\langle Z(\boldsymbol{\lambda}), [M]\rangle \in \mathbb{R}_{<0}$, while $\langle z, [M]\rangle$ lies in the open lower halfplane: the latter fact follows from the order one vanishing statement in Proposition 1(b). Thus the phase of M with respect to $\tilde{z}$ is in $(1, 2)$, so $M[-1]$ is in the heart of the t-structure since its phase with respect to $\tilde{z}$ is in $(0, 1)$.

Or else $\tilde{s}_\alpha^{-1}(M) = \widetilde{M}$ lies in the heart of τ_A, it is stable in stability $\tilde{z}$ and has phase in $(0, 1)$. To see this recall that $\langle Z(\boldsymbol{\lambda}), [M]\rangle$ lies in the open upper halfplane, while $\langle Z(\boldsymbol{\lambda}), [L]\rangle \in \mathbb{R}_{<0}$; since every nonzero proper subobject N in $\widetilde{M}$ contains M we see that

$$\langle Z(\boldsymbol{\lambda}), [N]\rangle = \langle Z(\boldsymbol{\lambda}), [\widetilde{M}]\rangle + s$$

for some $s \in \mathbb{R}_{>0}$. Thus the phase of N is smaller than the phase of $\widetilde{M}$, i.e., $\widetilde{M}$ is stable. It follows that the same is true for stability conditions in a neighborhood of $\tilde{z}$. □

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