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MINIMAL LIOUVILLE GRAVITY FROM DOUGLAS STRING EQUATION

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To Borya Feigin, whose contribution to Conformal Field Theory and Minimal Liouville Gravity is so great

Abstract. We describe the connection between Minimal Liouville gravity, Douglas string equation and Frobrenius manifolds. We show that the appropriate solution of the Douglas equation and a proper transformation from the KdV to the Liouville frames leads to the fulfilment of the selection rules of the underlying conformal field theory. We review the properties of Minimal Liouville gravity and Frobenius manifolds and show that the required solution of the string equation takes simple form in the flat coordinates on the Frobenious manifold in the case of unitary Minimal Liouville gravity.

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1. Introduction

The Minimal Liouville Gravity (MLG) is an interesting example of the noncritical String theory [18]. The matter sector of MLG is represented by (p, q) Minimal Model of CFT [2]. One of the main problems of MLG is the problem of calculating the correlation numbers of the so called physical observables.

In this article we review recent development of the approach to MLG based on the Douglas string equation. This equation has been initially introduced by Douglas [8] in the context of Matrix Models [5, 9, 11–14, 17, 19] to 2D gravity. The basic conjecture in this approach is formulated as follows. There exists some special solution of the String equation which is related to the generating function of the correlation numbers in MLG. So, the main question is how to find the appropriate solution of the Douglas string equation and the transformation from the parameters of the string equation to the coupling constants of MLG. Recently significant progress in the analysis of this problem has been achieved in the works [1, 3, 4]. In this paper we review these achievements, we describe the general procedure of

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calculating the correlation functions and demonstrate the results for the unitary series of MLG models.

The paper is organized as follows. In Section 2 we review the basic facts about the Minimal Liouville gravity. In Section 3 we recall briefly the notion of the Frobenius manifold and discuss its basic properties. In Section 4 we discuss the Frobenius manifold that appears in the context of Minimal Liouville Gravity. Section 5 is devoted to the connection between the Frobenius manifold structures, Integrable structures and the Douglas string equation. In Section 6 we focus on the scale invariance of the String equation in A_{q-1} case and in the Unitary models of MLG and discuss the problem of the resonance transformations. Here the main conjecture of the approach is formulated. Section 7 is devoted to one-point functions. The appropriate solution of the Douglas string equation is also discussed here. In Section 8 we consider the two-point functions and the exact expressions of the first-order resonances given in terms of Jacobi polynomials. We show that the special choice of the solution of the String equation together with the explicitly found resonance transformation ensure fulfilling the necessary selection rules for the correlation numbers in $(p,\, q)$ MLG.

2. Minimal Liouville Gravity

The Minimal Liouville gravity consists of Liouville theory of the scalar field ϕ, the matter sector which is taken to be a $(q,\, p)$ Minimal Model of CFT, and the ghost system.

2.1. Minimal models of CFT. The Minimal Model $\mathcal{M}_{q,p}$ has primary fields which correspond to integrable representations of Virasoro algebra and which are enumerated by elements of the Kac table: $\Phi_{m,n}$, where $m = 1, \ldots, q-1$ and $n = 1, \ldots, p-1$. Only half of the fields $\Phi_{m,n}$ are independent

$$\Phi_{m,n} = \Phi_{q-m,p-n}. \tag{2.1}$$

The operator product expansion (OPE) for these fields is the subject of the following fusion rules

$$[\Phi_{m_1,n_1}][\Phi_{m_2,n_2}] = \sum_{m=|m_1-m_2|:2}^{I(m_1,m_2)} \sum_{n=|n_1-n_2|:2}^{I(n_1,n_2)} [\Phi_{m,n}], \tag{2.2}$$

where $[\Phi_{m,n}]$ denotes the contribution of the irreducible Virasoro representation with the highest state $\Phi_{m,n}$. The summation goes with the step 2 and

$$I(a,\, b) = \min(a+b-1,\, 2q-a-b-1). \tag{2.3}$$

The small conformal group and OPE give strong constraints on the correlation functions. The constraints for one- and two-point correlation functions are

$$\langle \Phi_{m,n}(x) \rangle = 0, \tag{2.4}$$

$$\langle \Phi_{m_1,n_1}(x_1)\Phi_{m_2,n_2}(x_2) \rangle = 0, \quad m_1,\, n_1 \neq m_2,\, n_2. \tag{2.5}$$

For higher correlation numbers we also find a restriction which follows from the OPE fusion rules. For instance the three-point correlation functions satisfy

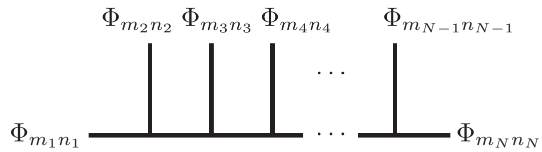
$$\langle \Phi_{m_1,n_1}\Phi_{m_2,n_2}\Phi_{m_3,n_3} \rangle = 0, \tag{2.6}$$

for

$$m_3 > I(m_1, m_2) = \begin{cases} m_1 + m_2 - 1, & m_1 + m_2 - 1 \leqslant q - 1, \\ 2q - m_1 - m_2 - 1, & m_1 + m_2 > q, \end{cases} \tag{2.7}$$

where we assume that $m_1 \leqslant m_2 \leqslant m_3$.

The following graphical representation allows to formulate these restrictions for the general correlation function



Here the external lines represent the primary fields in the correlation function

$$\langle \Phi_{m_1n_1} \Phi_{m_2n_2} \ldots \Phi_{m_Nn_N} \rangle. \tag{2.8}$$

From the fusion rules it follows that the correlator has to be zero if we cannot assign some set of pairs (k_i, l_i) to the internal lines, in such a way that in each vertex of the graph the following condition for the three pairs corresponding to the lines connected to this vertex is fulfilled for any permutation of the pairs

$$|m_1 - m_2| + 1 \leqslant m_3 \leqslant \min\{m_1 + m_2 - 1, 2q + 1 - m_1 - m_2\}, \tag{2.9}$$

$$|n_1 - n_2| + 1 \leqslant n_3 \leqslant \min\{n_1 + n_2 - 1, 2p + 1 - n_1 - n_2\}. \tag{2.10}$$

These equations represent the so called selection rules.

2.2. Liouville Field Theory. The Polyakov's continuous approach to two-dimensional quantum gravity [18] is defined through the path integral over two-dimensional Riemann metrics $g_{\mu\nu}$ interacting with some conformal matter.

Because of the conformal anomaly, in the conformal gauge $g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}$ it leads to the Liouville action

$$S_L = \frac{1}{4\pi} \int_M \sqrt{\hat{g}} \big(\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + Q\hat{R}\phi + 4\pi\mu e^{2b\phi}\big)\, d^2x, \tag{2.11}$$

where $\hat{g}_{\mu\nu}$ is some fixed background metric, μ is the cosmological constant and parameters Q, b are related to the central charge c_L of the Liouville theory

$$c_L = 1 + 6Q^2, \quad Q = b + b^{-1}. \tag{2.12}$$

The central charge c_M of the conformal matter is related to the central charge of the Liouville theory by the requirement of the cancellation of the Weyl anomaly

$$c_L + c_M = 26. \tag{2.13}$$

In the case of (q, p) Minimal Liouville Gravity, where the conformal matter is (q, p) Minimal Model of CFT, we find $b = \sqrt{\frac{q}{p}}$.

2.3. Correlation numbers of MLG. The observables of the (q, p) MLG are cohomologies of the BRST operator. They are in one-to-one correspondence with the primary fields in the Minimal Model of the matter sector. We denote them $O_{m,n}$. Explicitly,

$$O_{m,n} = \int_{x\in M} \mathcal{O}_{m,n}(x), \quad \mathcal{O}_{m,n}(x) = \Phi_{m,n}(x) e^{2b\delta_{m,n}\phi(x)}\sqrt{\hat{g}}\, d^2x. \tag{2.14}$$

The operators $O_{m,n}$ satisfy the same selection rules as $\Phi_{m,n}$. Moreover, they have the following scaling property

$$O_{m,n} \sim \mu^{-\delta_{m,n}}, \quad \delta_{m,n} = \frac{p+q-|pm-qn|}{2q}. \tag{2.15}$$

The correlation numbers in Minimal Liouville Gravity are defined as

$$Z_{m_1 n_1 \ldots m_N n_N} = \langle O_{m_1,n_1} \ldots O_{m_N,n_N} \rangle. \tag{2.16}$$

The generating function of these correlation numbers is

$$Z_L(\{\lambda_{m,n}\}) = \left\langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \right\rangle. \tag{2.17}$$

We note that it is a quasihomogeneous function, i.e.,

$$Z_L(\{\rho^{\delta_{m,n}}\lambda_{m,n}\}) = \rho^{\frac{p+q}{q}} Z_L(\{\lambda_{m,n}\}). \tag{2.18}$$

2.4. 3- and 4-point functions in MLG. The explicit formulae for two-, three- and four-point correlation numbers in MLG are known from the direct computations in the continuous approach. We can compare these results with those obtained from the approach based on the Douglas string equation.

To perform the comparison we construct the quantities which do not depend on the normalizations of the operators and the correlators

$$\frac{\langle\langle O_{m_1,n_1} O_{m_2,n_2} O_{m_3,n_3} \rangle\rangle^2}{\prod_{i=1}^3 \langle\langle O^2_{m_i,n_i} \rangle\rangle} = \frac{\prod_{i=1}^3 |m_i p - n_i q|}{p(p+q)(p-q)}, \tag{2.19}$$

$$\frac{\langle\langle O_{m_1,n_1} O_{m_2,n_2} O_{m_3,n_3} O_{m_4,n_4} \rangle\rangle}{\left(\prod_{i=1}^4 \langle\langle O^2_{m_i,n_i} \rangle\rangle\right)^{1/2}} = \frac{\prod_{i=1}^4 |m_i p - n_i q|}{2p(p+q)(p-q)} \times$$
$$\times \left(\sum_{i=2}^{4} \sum_{r=-(m_1-1)}^{m_1-1} \sum_{t=-(n_1-1)}^{n_1-1} |(m_i - r)p - (n_i - t)q| - m_1 n_1 (m_1 p + n_1 q) \right), \tag{2.20}$$

where $\langle\langle \ldots \rangle\rangle = \frac{\langle \ldots \rangle}{\langle 1 \rangle}$. The summations over r, t in the last formula go with the step 2.

2.5. Contact terms. The correlation numbers involve integration over n points on the 2D surface

$$Z_{m_1 n_1 \ldots m_N n_N} = \int \langle \mathcal{O}_{m_1,n_1}(x_1) \ldots \mathcal{O}_{m_N,n_N}(x_N) \rangle\, d^2x_1 \ldots d^2x_{N-3}. \tag{2.21}$$

The contact delta-like terms may appear when two or more points x_i are coincident. The ambiguity in contact terms leads to the fact that we can add to the n-point correlation numbers some k-point correlation numbers ($k < n$). For example,

$$\langle O_{m_1,n_1} O_{m_2,n_2} \rangle \to \langle O_{m_1,n_1} O_{m_2,n_2} \rangle + \sum_{m,n} A_{m,n}^{(m_1 n_1)(m_2 n_2)} \langle O_{m,n} \rangle. \tag{2.22}$$

This substitution is equivalent to the change of coupling constants in the generating function

$$\lambda_{m,n} \to \lambda_{m,n} + \sum_{m_1,n_1,m_2,n_2} A_{m,n}^{(m_1 n_1)(m_2 n_2)} \lambda_{m_1 n_1} \lambda_{m_2 n_2}. \tag{2.23}$$

In MLG there is some restriction on this change of coupling constants since they have certain scaling dimensions

$$\lambda_{m,n} \sim \mu^{\delta_{m,n}}. \tag{2.24}$$

Therefore we can demand that all the terms have the same dimensions

$$\delta_{m,n} = \delta_{m_1,n_1} + \delta_{m_2,n_2} + \delta_{m_3,n_3} + \dots. \tag{2.25}$$

The addition of the contact terms is equivalent to some non-linear (polynomial) change of the coupling constants

$$\lambda_{m,n} \to A\mu^{\delta_{m,n}} + \sum_{m_1,n_1} C_{m,n}^{(m_1 n_1)} \mu^{\delta_{m,n} - \delta_{m_1,n_1}} \lambda_{m_1,n_1} \tag{2.26}$$

$$+ \sum_{m_1,n_1} \sum_{m_2,n_2} C_{m,n}^{(m_1 n_1)(m_2 n_2)} \mu^{\delta_{m,n} - \delta_{m_1,n_1} - \delta_{m_2,n_2}} \lambda_{m_1 n_1} \lambda_{m_2 n_2} + \dots. \tag{2.27}$$

Only the terms in these sums with the integer and positive degrees of

$$\delta_{m,n} - \delta_{m_1,n_1} - \delta_{m_2,n_2} - \dots, \tag{2.28}$$

have nonvanishing coefficients. So, there exist different "systems of coordinates" on the space of the parameters $\lambda_{m,n}$, and, in general case, the MLG coordinate frame does not coincide with the coordinate system which is natural for the Douglas string equation. The change of the variables conserves the property of the quasi-homogeneity

$$Z_L(\{\rho^{\delta_{m,n}} \lambda_{m,n}\}) = \rho^{\frac{p+q}{q}} Z_L(\{\lambda_{m,n}\}). \tag{2.29}$$

3. Frobenius Manifolds

A commutative and associative algebra A with unity, equipped with a nondegenerate invariant pairing (,) is called Frobenius algebra. The invariance means that for any three vectors a, b, c in A:

$$(a \cdot b,\, c) = (a,\, b \cdot c). \tag{3.1}$$

Let M be n-dimensional manifold with the flat metric $\eta_{\alpha\beta}\, dv^\alpha dv^\alpha$ which is constant in the flat coordinates v^α.

We introduce the structure of the Frobenius algebra in the tangent space $T_{\mathbf{v}}M$ by means of the following identification of the bases

$$\frac{\partial}{\partial v^\alpha} \to e_\alpha, \tag{3.2}$$

Thus, we can multiply tangent vectors at any point of M

$$e_\alpha e_\beta = C^\gamma_{\alpha\beta} e_\gamma. \tag{3.3}$$

The structure constants $C^\gamma_{\alpha\beta}$ may depend on v^α. Such manifold M can be called quasi-Frobenius manifold.

Definition 3.1. The manifold M is called Frobenius manifold if the two structures described above are adjusted with each other in such a way that

(1) the bilinear form $(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta})$ of the Frobenius algebra is identical to the metric $\eta_{\alpha\beta}$;
(2) the structure of the Frobenius algebra at each point of M and the metric on M are connected by the following relation

$$\nabla_\rho C_{\alpha\beta\gamma} = \nabla_\alpha C_{\rho\beta\gamma}. \tag{3.4}$$

This relation is equivalent to the requirement that there exists a function F on M which is connected with the structure constants of the Frobenius algebra as follows

$$C_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}, \tag{3.5}$$

where

$$C_{\alpha\beta\gamma} = \eta_{\alpha\rho} C^\rho_{\beta\gamma}. \tag{3.6}$$

The function F is called Frobenius potential. The consistency of this property with the associativity of the Frobenius algebra is known as WDVV condition [7]

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\rho}\, \eta^{\rho\lambda}\, \frac{\partial^3 F}{\partial v^\lambda \partial v^\mu \partial v^\nu} = \frac{\partial^3 F}{\partial v^\nu \partial v^\beta \partial v^\rho}\, \eta^{\rho\lambda}\, \frac{\partial^3 F}{\partial v^\lambda \partial v^\mu \partial v^\alpha}. \tag{3.7}$$

The following statement [10] follows from these properties of the Frobenius manifold M. There exist one-parametric flat deformation $\widetilde{\nabla}_\alpha$ of the connection ∇_α

$$\widetilde{\nabla}_\alpha x^\gamma = \nabla_\alpha x^\gamma + z C^\gamma_{\alpha\beta} x^\beta, \tag{3.8}$$

or, equivalently,

$$[\widetilde{\nabla}_\alpha(z), \widetilde{\nabla}_\beta(z)] = 0. \tag{3.9}$$

The proof is based on the associativity of the Frobenius algebra and the equation (3.4). As a consequence of (3.9), there exist n linear independent solutions

$$\theta^\alpha(v, z) = \sum_{k=0}^{\infty} \theta^\alpha_k(v) z^k, \tag{3.10}$$

of the equation $\widetilde{\nabla}_\alpha d\theta^\lambda(v, z) = 0$.

The equation (3.9) is equivalent to the following recursion relations

$$\frac{\partial^2 \theta^\lambda_{k+1}}{\partial v^\alpha \partial v^\beta}(z) = C^\gamma_{\alpha\beta} \frac{\partial \theta^\lambda_k}{\partial v^\gamma}(z). \tag{3.11}$$

The functions $\theta^\alpha(v,\,z)$ can be considered as the flat coordinates of the deformed connection $\widetilde{\nabla}_\alpha(z)$. We can choose $\theta^\lambda(v,\,z)$ in such a way that $\theta^\lambda(v,\,0)=\theta_0^\lambda(v)=v^\lambda$.

4. A_{q-1} Frobenius Manifold

Let $Q(y)$ be a polynomial of y

$$Q(y)=y^q+u_1y^{q-2}+\ldots+u_{q-1}, \tag{4.1}$$

where $\{u_\alpha\}$ represent some coordinates on M. We call $\{u_i\}$ the canonical coordinates. A_{q-1} Frobenius algebra is the space of polynomials of y modulo polynomial $\frac{dQ}{dy}$:

$$A_{q-1}(u)=\mathbb{C}[y]/\frac{dQ}{dy}. \tag{4.2}$$

The corresponding manifold M is called the Frobenius manifold of A_{q-1} type. The polynomials

$$P_i(y)=\frac{\partial Q}{\partial u_i}, \tag{4.3}$$

form a basis in the tangent space $T_{\mathbf{v}}M$. An invariant bilinear form is defined by

$$(P_1,\,P_2)=\operatorname*{res}_{y=\infty}\left(\frac{P_1(y)P_2(y)}{\frac{dQ}{dy}(y)}\right). \tag{4.4}$$

With this definition one can verify that the corresponding metric is flat and

$$C_{\alpha\beta\gamma}=\nabla_\alpha\nabla_\beta\nabla_\gamma F(u). \tag{4.5}$$

To this end we perform the transformation from the canonical $\{u_\alpha\}$ to the new coordinates $\{v_\alpha\}$ by means of the following relation

$$y=z-\frac{1}{q}\left(\frac{v_1}{z}+\frac{v_2}{z^2}+\cdots+\frac{v_{q-1}}{z^{q-1}}\right)+\mathcal{O}\left(\frac{1}{z^{q+1}}\right), \tag{4.6}$$

where $z^q=Q(y)$.

We note the following useful properties of the flat coordinates. In the flat coordinates

$$\eta^{\alpha\beta}=\left(\frac{\partial Q}{\partial v_\alpha},\,\frac{\partial Q}{\partial v_\beta}\right)=\delta_{\alpha+\beta,n+1}, \tag{4.7}$$

$$C_{\alpha\beta\gamma}=-q\operatorname*{res}_{y=\infty}\left(\frac{\frac{\partial Q}{\partial v^\alpha}\frac{\partial Q}{\partial v^\beta}\frac{\partial Q}{\partial v^\gamma}}{\frac{dQ}{dy}}\right)=\frac{\partial^3F}{\partial v^\alpha\partial v^\beta\partial v^\gamma}, \tag{4.8}$$

$$\theta_{\alpha,k}=-c_{\alpha,k}\operatorname*{res}_{y=\infty}Q^{k+\frac{\alpha}{q}}(y), \tag{4.9}$$

where

$$c_{\alpha,k}^{-1}=\left(\frac{\alpha}{q}\right)_{k+1}\quad\text{and}\quad(a)_{q-1}=\frac{\Gamma(a+q-1)}{\Gamma(a)}. \tag{4.10}$$

5. Frobenius Manifolds and String Equation

We define a function $S(v, t)$ on M which depends on the additional parameters $\{t_k^\alpha\}$

$$S(v,\, t_k^\alpha) = \sum_{\alpha=1}^{n} \sum_{k\geqslant 0} t_k^\alpha \theta_{\alpha,k}(v). \tag{5.1}$$

The equation

$$\frac{\partial S}{\partial v^\alpha} = 0, \tag{5.2}$$

is called the string equation. In the case of the Frobenius manifold of A_{q-1}-type it is nothing but the Douglas string equation.

We define the function $Z[t] = \log \tau(t)$, where

$$Z[t] = \frac{1}{2} \int_0^{v=v^*(t)} \Omega, \tag{5.3}$$

and

$$\Omega = C_\alpha^{\beta\gamma}(v) \frac{\partial S(v,\, t)}{\partial v^\beta} \frac{\partial S(v,\, t)}{\partial v^\gamma} dv^\alpha, \tag{5.4}$$

is the differential form. The upper limit $v^*(t)$ in (5.3) is one of the solutions of the string equation (5.2). Taking into account the associativity of the algebra A_{q-1} and the equations (3.11), we can prove that Ω is closed one-form. One can also show that Z(t) satisfy

$$\frac{\partial^2 Z(t)}{\partial t_k^\alpha \partial t_0^1} = \theta_{\alpha,k}(v(t)). \tag{5.5}$$

In particular,

$$v^\alpha(t) = \eta^{\alpha\beta} \frac{\partial^2 Z}{\partial t_0^\beta \partial t_0^1}, \tag{5.6}$$

and for $v^{q-1}(t) = u_1(t)$

$$\frac{\partial^2 Z}{\partial x^2} = u_1(t). \tag{5.7}$$

Taking into account $\theta_{\alpha,0} = v_\alpha$ we obtain that

$$\frac{\partial^2 Z(t)}{\partial t_0^\alpha \partial t_0^1} = v_\alpha. \tag{5.8}$$

Since Z satisfy equations (5.2) and (5.7), it is a tau-function of the integrable hierarchy connected with the corresponding Frobenius manifold.

$Z[\{t_k^\alpha\}]$ is the logarithm of the tau-function given by

$$Z[\{t_k^\alpha\}] = \frac{1}{2} \int_0^{v^*(t)} C_\alpha^{\beta\gamma} \frac{\partial S}{\partial v^\beta} \frac{\partial S}{\partial v^\gamma} dv^\alpha, \tag{5.9}$$

where $v^*(t)$ is the solution of the string equation

$$\left. \frac{\partial S}{\partial v^\alpha} \right|_{v^\alpha = v^{*\,\alpha}(t)}, \tag{5.10}$$

and

$$S = \sum_{\alpha=1}^{q-1} \sum_{k} t_k^{\alpha} \theta_{\alpha,k}. \tag{5.11}$$

6. Scale Invariance in A_{q-1} Case and $(q+1,\, q)$ MLG

In what follows we consider the unitary models of MLG (see [6], [4]).

Let us take the action in the form

$$S = \operatorname*{res}_{y=\infty} \left[Q^{\frac{q+2}{q}} + \sum_{m,n}^{(q+1)m-qn>0} \tau_{mn} Q^{\frac{(q+1)m-qn}{q}} \right], \tag{6.1}$$

It is easy to check that $Q[y,\, u_\alpha]$ and $S[u_\alpha,\, \tau_{mn}]$ are quasi-homogeneous functions

$$Q[\rho y,\, \rho^{r_\alpha} u_\alpha] = \rho^q Q[y,\, u_\alpha], \quad S[\rho^{r_\alpha} u_\alpha,\, \rho^{\sigma_{mn}} \tau_{mn}] = \rho^{q+1} S[u_\alpha,\, \tau_{mn}]. \tag{6.2}$$

Here we denote

$$r_\alpha = q - \alpha - 1, \quad \sigma_{mn} = 2q + 1 - |(q+1)m - qn|. \tag{6.3}$$

We call $\{\sigma_{mn}\}$ the set of the scaling indices of the set $\{\tau_{mn}\}$. It was found in [8] that the numbers $\delta_{mn} = \frac{\sigma_{mn}}{2q}$ coincide with the gravitational dimensions of the physical fields in $(q+1,\, q)$ unitary Minimal Liouvillle gravity [15]. The function $Z[\tau_{mn}]$ is a quasi-homogeneous function

$$Z[\rho^{2q\delta_{mn}} \tau_{mn}] = \rho^{2q+1} Z[\tau_{mn}]. \tag{6.4}$$

The following relation

$$\sigma_{mn} = \sigma_{k_1 l_1} + \sigma_{k_2 l_2} + \ldots + \sigma_{k_N l_N}, \tag{6.5}$$

is known as a resonance condition. A transformation $\tau_{mn} \to \lambda_{mn}$ of the form

$$\begin{aligned} \tau_{mn} = \lambda_{mn} &+ \sum_{k_1,l_1,k_2,l_2} A_{mn}^{k_1 l_1; k_2, l_2} \lambda_{k_1,l_1} \lambda_{k_2,l_2} \\ &+ \sum_{k_1,l_1,k_2,l_2,k_3,l_3} A_{mn}^{k_1 l_1; k_2, l_2; k_3, l_3} \lambda_{k_1,l_1} \lambda_{k_2,l_2} \lambda_{k_3,l_3} + \ldots, \end{aligned} \tag{6.6}$$

is called the resonance transformation if (6.5) is satisfied for each term.

Now we are in the position to formulate our main conjecture. There exist a solution of the string equation and a choice of the resonance transformation such that the function

$$\begin{aligned} Z_L[\{\lambda_{mn}\}] &= \left\langle \exp \sum_{m,n} \lambda_{m,n} O_{m,n} \right\rangle \\ &= \sum_{N=0}^{\infty} \sum_{m_i,n_i} \frac{\lambda_{m_1 n_1} \ldots \lambda_{m_N n_N}}{n!} \langle O_{m_1 n_1} \ldots O_{m_N n_N} \rangle \end{aligned} \tag{6.7}$$

is the generating function of the correlators in the Minimal Liouville Gravity. In particular, all correlators $\langle O_{m_1 n_1} \ldots O_{m_N n_N} \rangle$ are zero if they are forbidden by the conformal fusion rules.

After performing the resonance transform

$$t_{mn} = \lambda_{mn} + A_{mn}\mu^{\delta_{mn}} + \sum_{m_1,n_1}^{\delta_{m_1n_1}\leqslant\delta_{mn}} A_{mn}^{m_1n_1}\mu^{\delta_{mn}-\delta_{m_1n_1}}\lambda_{m_1n_1} + \sum_{m_1,n_1,m_2,n_2}^{\delta_{m_1n_1}+\delta_{m_2n_2}\leqslant\delta_{mn}} A_{mn}^{m_1n_1,m_2n_2}\mu^{\delta_{mn}-\delta_{m_1n_1}-\delta_{m_2n_2}}\lambda_{m_1n_1}\lambda_{m_2n_2} + \dots, \quad (6.8)$$

the action is written in the form

$$S_L[v_\alpha,\, \{\lambda_{mn}\}] = S^{(0)}[v_\alpha] + \sum_{m,n}\lambda_{mn}\, S^{(mn)}(v_\alpha) + \sum_{m_1,n_1,m_2,n_2}\lambda_{m_1n_1}\lambda_{m_2n_2}\, S^{(m_1n_1,m_2n_2)}(v_\alpha) + \dots. \quad (6.9)$$

From (4.9) and (12.1) we obtain

$$S^{(0)} = \operatorname*{res}_{y=\infty}\bigl[Q^{\frac{2q+1}{q}} + \mu\, Q^{\frac{1}{q}}\bigr], \quad (6.10)$$

$$S^{(mn)} = \operatorname*{res}_{y=\infty}\left[Q^{\frac{(q+1)m-qn}{q}} + \sum_{l=2}^{m} A_{ml}^{mn}\,\mu^{\frac{l}{2}}\, Q^{\frac{(q+1)m-ql}{q}}\right], \quad (6.11)$$

where A_{kl}^{mn} are the coefficients of the resonance relations. The higher coefficients can be also easily written in terms of the coefficients $A_{kl}^{\{m_in_i\}}$.

The generating function is given by

$$Z_L[\{\lambda_{mn}\}] = \frac{1}{2}\int_0^{\mathbf{v}^*} C_\alpha^{\beta\gamma}(v)\frac{\partial S_L}{\partial v^\beta}\frac{\partial S_L}{\partial v^\gamma}dv^\alpha, \quad (6.12)$$

where $\mathbf{v}^*$ is defined as a function of the parameters $\{\lambda_{mn}\}$ from the Douglas string equation (5.2).

7. One-Point Functions

To compute the one-point function which is given by the integral

$$\langle O_{mn}\rangle = \int_0^{v_\alpha^0} C_{\beta\gamma}^{\alpha}\frac{\partial S^{(0)}}{\partial v^\beta}\frac{\partial S^{(mn)}}{\partial v^\gamma}dv_\alpha, \quad (7.1)$$

we need to know the upper limit v_α^0 which is the solution of the string equation for all couplings (except $\lambda_{11} = \mu$) equal to zero

$$v_\alpha^0 = v_\alpha^*(\lambda_{mn})\big|_{\lambda_{mn}=0,\lambda_{11}=\mu}. \quad (7.2)$$

That is, v_α^0 satisfy

$$\left.\frac{\partial S^{(0)}}{\partial v_\mu}\right|_{v_\alpha=v_\alpha^0} = 0. \quad (7.3)$$

We can write $S^{(0)}$ and $S^{(mn)}$ in the following explicit form

$$S^{(0)} = -\frac{\theta_{1,2}}{c_{1,2}} - \mu\frac{\theta_{1,0}}{c_{1,0}}, \tag{7.4}$$

$$S^{(mn)} = -\frac{\theta_{m,m-n}}{c_{m,m-n}} - \sum_{l=2:2}^{m} A_{ml}^{mn}\mu^{\frac{l}{2}}\frac{\theta_{m,m-l}}{c_{m,m-l}}. \tag{7.5}$$

In [4] it was shown that on the line $v_{i>1}=0$,

$$\begin{cases} k \text{ even:} & \frac{\partial\theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,\alpha}\, x_{\lambda,k}\left(-\frac{v_1}{q}\right)^{\frac{k}{2}q}, \\ k \text{ odd:} & \frac{\partial\theta_{\lambda,k}}{\partial v_\alpha} = \delta_{\lambda,q-\alpha}\, y_{\lambda,k}\left(-\frac{v_1}{q}\right)^{\frac{k-1}{2}q+\lambda}, \end{cases} \tag{7.6}$$

where

$$x_{\lambda,k} = \frac{1}{\left(\frac{\lambda}{q}\right)_{\frac{k}{2}}\left(\frac{k}{2}\right)!} \quad \text{and} \quad y_{\lambda,k} = -\frac{1}{\left(\frac{\lambda}{q}\right)_{\frac{k+1}{2}}\left(\frac{k-1}{2}\right)!}. \tag{7.7}$$

Using this statement we find that the string equation has the solutions of the form $v_\alpha^0 = 0$ for $\alpha \neq 1$. The coordinate v_1^0 itself is a root of the equation

$$\frac{\partial S^{(0)}}{\partial v_1} = 0 \tag{7.8}$$

where after taking derivative we set all v_α for $\alpha \neq 1$ to zero.

In what follows we use the following result [4] for the structure constant in the flat coordinates on the line $v_{\alpha>0} = 0$

$$C_{\alpha\beta\gamma} = \Theta_{1,q-1}(\alpha+\beta-\gamma)\left(-\frac{v_1}{q}\right)^{\frac{\alpha+\beta+\gamma-q-1}{2}} \text{ if } \frac{\alpha+\beta+\gamma-q-1}{2} \in \mathbb{N}, \text{ otherwise } 0. \tag{7.9}$$

for $\alpha \geqslant \beta \geqslant \gamma$. Here $\Theta_{A,B}(x) = 1$ if $x \in [A,\, B]$ and is zero otherwise. In particular, $C_\alpha^{q-1,\beta} = \delta_{\alpha\beta}$. Taking into account (7.6) we find

$$\langle O_{mn}\rangle = \int_0^{v_1^0} \frac{\partial S^{(0)}}{\partial v_1}\frac{\partial S^{(mn)}}{\partial v_1} dv_1. \tag{7.10}$$

From (7.6) it follows that the one-point function is not equal to zero only if one of the following two conditions is satisfied

$$1)\, m = 1 \text{ mod } q, \quad (m-n) \text{ is even}, \tag{7.11}$$

$$2)\, m = q-1 \text{ mod } q, \quad (m-n) \text{ is odd}. \tag{7.12}$$

In the first case we have the only possible pair $(m,\, n) = (1,\, 1)$ which corresponds to the unit operator. In the second case we find from the following expression for the gravitational dimension of the one-point function

$$[\langle O_{mn}\rangle] = \frac{2q+1}{q} - \delta_{mn} = \frac{m-n+2}{2} + \frac{m+1}{2q}, \tag{7.13}$$

that the one-point function is analytic in μ and, therefore, should not be considered [1].

We arrive to the conclusion that the special solution of String equation considered above ensure the requirements of the selection rules for the one-point functions.

8. Two-Point Functions

We are now going to consider the two-point function. From (6.12) we find

$$\langle O_{m_1 n_1} O_{m_2 n_2}\rangle = \sum_{\gamma=1}^{q-1}(-q)^{1-\gamma}\int_0^{v_1^0} dv_1\, v_1^{\gamma-1}\,\frac{\partial S^{(m_1 n_1)}}{\partial v_\gamma}\,\frac{\partial S^{(m_2 n_2)}}{\partial v_\gamma}. \tag{8.1}$$

It follows from (7.6) that $\frac{\partial S^{(mn)}}{\partial v^\gamma} \neq 0$ if one of the following two conditions is satisfied

$$\begin{aligned} &1)\ \ \gamma = m \bmod q \quad \text{and} \quad (m-n) \text{ is even,}\\ &2)\ \ \gamma = q - m \bmod q \quad \text{and} \quad (m-n) \text{ is odd.}\end{aligned} \tag{8.2}$$

If the first pair (m_1, n_1) satisfies the first condition while the second pair (m_2, n_2) is subject of the second condition and vice versa, then we get the regular expression for the two-point function. Hence we are left with the two options where both pairs satisfy either the first or the second condition in (8.2).

If both $(m - n_1)$ and $(m - n_2)$ are even we find the following requirement for two-point functions

$$\langle O_{mn_1} O_{mn_2}\rangle = \int_0^{v_1^0} dv_1\, v_1^{m-1}\,\frac{\partial S^{(mn_1)}}{\partial v_m}\,\frac{\partial S^{(mn_2)}}{\partial v_m} = 0 \quad \text{if } n_1 \neq n_2. \tag{8.3}$$

Making the substitution

$$t = 2\left(\frac{v_1}{v_1^0}\right)^q - 1, \tag{8.4}$$

and denoting

$$\frac{\partial S^{(mn)}}{\partial v_m} = L_{\frac{m-n}{2}}(t), \tag{8.5}$$

we find the following consequence of the diagonality condition for the two-point correlation function:

$$\langle O_{mn_1} O_{mn_2}\rangle = \int_{-1}^1 dt\,(1+t)^{\frac{m-q}{q}}\, L_{\frac{m-n_1}{2}}(t) L_{\frac{m-n_2}{2}}(t) = 0 \quad \text{if } n_1 \neq n_2. \tag{8.6}$$

Hence, the selection rules for the two-point correlation numbers requires that the polynomials $L_{\frac{m-n}{2}}$ form an orthogonal set of Jacobi polynomials

$$\frac{\partial S^{(mn)}}{\partial v_m} = P_{\frac{m-n}{2}}^{(0,\frac{m-q}{q})}(t). \tag{8.7}$$

In the second case when both $(m - n_1)$ and $(m - n_2)$ are odd we have

$$\langle O_{mn_1} O_{mn_2}\rangle = \int_0^{v_1^0} dv_1\, v_1^{q-m-1}\,\frac{\partial S^{(mn_1)}}{\partial v_{q-m}}\,\frac{\partial S^{(mn_2)}}{\partial v_{q-m}} = 0 \quad \text{if } n_1 \neq n_2. \tag{8.8}$$

Denoting

$$\frac{\partial S^{(mn)}}{\partial v_{q-m}} = (1+t)^{\frac{m}{q}} L_{\frac{m-n-1}{2}}(t), \tag{8.9}$$

we find the following consequence of the diagonality condition for the two-point correlation function in this case:

$$\langle O_{mn_1} O_{mn_2} \rangle = \int_{-1}^{1} dt\,(1+t)^{\frac{m}{q}}\, L_{\frac{m-n_1-1}{2}}(t) L_{\frac{m-n_2-1}{2}}(t) = 0 \quad \text{if } n_1 \neq n_2. \tag{8.10}$$

It means that the polynomials $L_{\frac{m-n-1}{2}}$ form an orthogonal set of Jacobi polynomials

$$\frac{\partial S^{(mn)}}{\partial v_{q-m}} = (1+t)^{\frac{m}{q}} P^{(0,\frac{m}{q})}_{\frac{m-n-1}{2}}(t). \tag{8.11}$$

At last, inserting these explicit expressions for $\frac{\partial S^{(mn)}}{\partial v_1}$ to the equation (7.10) we arrive to the equation

$$\langle O_{1n} \rangle = \int_{-1}^{1} (1+t)^{\frac{1-q}{q}} \frac{\partial S^{(0)}}{\partial v_1}(t) P^{(0,\frac{1-q}{q})}_{\frac{1-n}{2}}(t)\, dt = 0. \tag{8.12}$$

for odd n greater than 1.

Taking into account this result we obtain the following explicit expression:

$$\frac{\partial S^{(0)}}{\partial v_1} = P_1^{(0,\frac{1-q}{q})}(t) - P_0^{(0,\frac{1-q}{q})}(t). \tag{8.13}$$

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