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SOME TRANSFORMATION FORMULAS ASSOCIATED WITH ASKEY–WILSON POLYNOMIALS AND LASSALLE'S FORMULAS FOR MACDONALD–KOORNWINDER POLYNOMIALS

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Dedicated to Boris Feigin on his 60th birthday

Abstract. We present a fourfold series expansion representing the Askey–Wilson polynomials. To obtain the result, a sequential use is made of several summation and transformation formulas for the basic hypergeometric series, including the Verma's q-extension of the Field and Wimp expansion, Andrews' terminating q-analogue of Watson's ${}_3F_2$ sum, Singh's quadratic transformation. As an application, we present an explicit formula for the Koornwinder polynomial of type BC_n $(n \in \mathbb{Z}_{>0})$ with one row diagram. When the parameters are specialized, we recover Lassalle's formula for Macdonald polynomials of type B_n, C_n and D_n with one row diagram, thereby proving his conjectures.

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1. Introduction

Let a, b, c, d, $q \in \mathbb{C}$ be parameters with the condition $|q| < 1$. Let D denote the Askey–Wilson q-difference operator [1]

$$D = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}(T_{q,x}^{+1}-1) + \frac{(1-a/x)(1-b/x)(1-c/x)(1-d/x)}{(1-1/x^2)(1-q/x^2)}(T_{q,x}^{-1}-1), \qquad (1.1)$$

where the q-shift operators are defined by $T_{q,x}^{\pm 1} f(x) = f(q^{\pm 1}x)$. Recall the fundamental facts about the Askey–Wilson polynomial $p_n(x; a, b, c, d\,|\,q)$ $(n \in \mathbb{Z}_{\geqslant 0})$. It is a symmetric Laurent polynomial in x and characterized by the two conditions: (1) $p_n(x)$ has the highest degree n, (2) $p_n(x)$ is an eigenfunction of the operator D.

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Askey–Wilson's celebrated formula reads [1]

$$p_n(x) = a^{-n}(ab,\, ac,\, ad;\, q)_n\, {}_4\phi_3 \left[\begin{matrix} q^{-n},\, abcdq^{n-1},\, ax,\, a/x \\ ab,\, ac,\, ad \end{matrix} ;\, q,\, q \right], \tag{1.2}$$

$$Dp_n(x) = \left(q^{-n} + abcdq^{n-1} - 1 - abcdq^{-1}\right) p_n(x). \tag{1.3}$$

Here and hereafter we use the standard notation (see [2] for more details)

$$(a_1,\, a_2,\, \ldots,\, a_k;\, q)_n = \prod_{j=1}^{k} \prod_{i=0}^{n-1} (1 - q^i a_j), \tag{1.4}$$

$${}_{r+1}\phi_r \left[\begin{matrix} a_1,\, a_2,\, \ldots,\, a_{r+1} \\ b_1,\, b_2,\, \ldots,\, b_r \end{matrix} ;\, q,\, x \right] = \sum_{m \geqslant 0} \frac{(a_1,\, a_2,\, \ldots,\, a_{r+1};\, q)_m}{(q,\, b_1,\, b_2,\, \ldots,\, b_r;\, q)_m} x^m. \tag{1.5}$$

Let $s \in \mathbb{C}^\times$ be a parameter. Introduce λ satisfying $s = q^{-\lambda}$. Then we have $T_{q,x}^{\pm 1} x^{-\lambda} = s^{\pm 1} x^{-\lambda}$. Let $f(x;\, s) = f(x;\, s\,|\,a,\, b,\, c,\, d\,|\,q)$ be a formal series in x

$$f(x;\, s) = x^{-\lambda} \sum_{n \geqslant 0} c_n x^n, \quad c_0 \neq 0, \tag{1.6}$$

satisfying the q-difference equation

$$Df(x;\, s) = \left(s + \frac{abcd}{qs} - 1 - \frac{abcd}{q} \right) f(x;\, s). \tag{1.7}$$

With the normalization $c_0 = 1$, equation (1.7) determines the coefficients $c_n = c_n(s\,|\,a,\, b,\, c,\, d\,|\,q)$ uniquely as rational functions in a, b, c, d, q and s.

By using (1.2), we can easily find an explicit formula for $f(x;\, s)$ (see Section 2, Theorem 2.1).

Theorem 1.1. *We have*

$$f(x;\, s) = x^{-\lambda} \frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty} \sum_{n \geqslant 0} \frac{(qs^2/a^2;\, q)_n}{(q;\, q)_n} (ax/s)^n$$
$$\times\, {}_6\phi_5 \left[\begin{matrix} q^{-n},\, q^{n+1}s^2/a^2,\, s,\, qs/ab,\, qs/ac,\, qs/ad \\ q^2s^2/abcd,\, q^{1/2}s/a,\, -q^{1/2}s/a,\, qs/a,\, -qs/a \end{matrix} ;\, q,\, q \right]. \tag{1.8}$$

Remark 1.1. When $\lambda = n \in \mathbb{Z}_{\geqslant 0}$, the series $f(x,\, q^{-n})$ must be proportional to the Askey–Wilson polynomial $p_n(x)$, in particular, indicating the termination of the series. Note that, however, such termination can not easily be seen from the expression (1.8). This is one of the reasons that we seek another expression below.

Studying the series $f(x;\, s)$, one finds that several interesting techniques are involved including the Verma's q-extension of the Field and Wimp expansion, Andrews' terminating q-analogue of Watson's ${}_3F_2$ sum, Singh's quadratic transformation [2] (see Section 3).

Definition 1.1. Set

$$c_e(k,\, l;\, s) = c_e(k,\, l;\, s\,|\,a,\, c\,|\,q)$$

and

$$c_o(m,\, n;\, s) = c_o(m,\, n;\, s\,|\,a,\, b,\, c,\, d\,|\,q)$$

by

$$c_e(k,\,l;\,s)=\frac{(a^2;\,q^2)_k(q^{4l}s^2;\,q^2)_k}{(q^2;\,q^2)_k(q^{4l}q^2s^2/a^2;\,q^2)_k}(q^2/a^2)^k \times\frac{(c^2/q;\,q^2)_l(s^2/a^2;\,q^2)_l}{(q^2;\,q^2)_l(q^3s^2/a^2c^2;\,q^2)_l}\frac{(s;\,q)_{2l}(q^2s^2/a^4;\,q^2)_{2l}}{(qs/a^2;\,q)_{2l}(s^2/a^2;\,q^2)_{2l}}(q^2/c^2)^l, \tag{1.9}$$

$$c_o(m,\,n;\,s)=\frac{(-b/a;\,q)_m(s;\,q)_m(qs/cd;\,q)_m(qs^2/a^2c^2;\,q)_m}{(q;\,q)_m(q^2s^2/abcd;\,q)_m(qs^2/a^2c^2;\,q^2)_m}(q/b)^m \times\frac{(-d/c;\,q)_n(q^ms;\,q)_n(qs/ab;\,q)_n(-q^mqs/ac;\,q)_n(q^mqs^2/a^2c^2;\,q)_n}{(q;\,q)_n(q^mq^2s^2/abcd;\,q)_n(-qs/ac;\,q)_n(q^{2m}qs^2/a^2c^2;\,q^2)_n}(q/d)^n. \tag{1.10}$$

Remark 1.2. (1) The 'even generators' $c_e(k,\,l;\,s)$ are basically composed in terms of the q-shifted factorials with the base q^2, and the 'odd ones' $c_o(k,\,l;\,s)$ are with the base q. (2) The $c_e(k,\,l;\,s)$ does not depend on b and d. (3) The $c_o(m,\,n;\,s)$ can be recast as

$$c_o(m,\,n;\,s)=\frac{(-b/a;\,q)_m(qs/cd;\,q)_m}{(q;\,q)_m(-qs/ac;\,q)_m} \times\frac{(s;\,q)_{m+n}(-qs/ac;\,q)_{m+n}(qs^2/a^2c^2;\,q)_{m+n}}{(q^2s^2/abcd;\,q)_{m+n}(q^{1/2}s/ac;\,q)_{m+n}(-q^{1/2}s/ac;\,q)_{m+n}}(q/b)^m \times\frac{(-d/c;\,q)_n(qs/ab;\,q)_n}{(q;\,q)_n(-qs/ac;\,q)_n}(q/d)^n. \tag{1.11}$$

Now we state our main result in the present paper (see Section 3, Theorems 3.1 and 3.2).

Theorem 1.2. *Let $s\in\mathbb{C}$ be generic. We have*

$$f(x;\,s)=x^{-\lambda}\sum_{k,l,m,n\geqslant 0}c_e(k,\,l;\,q^{m+n}s)c_o(m,\,n;\,s)x^{2k+2l+m+n}. \tag{1.12}$$

This gives us a fourfold summation formula for the Askey–Wilson polynomial.

Theorem 1.3. *Let $\lambda\in\mathbb{Z}_{\geqslant 0}$. We have the formula for the Askey–Wilson polynomial $p_\lambda(x)$ represented as a sum of monomials in x with factorized coefficients*

$$p_\lambda(x)=(abcdq^{\lambda-1};\,q)_\lambda\sum_{(k,l,m,n)\in\mathcal{P}_\lambda}c_e(k,\,l;\,q^{m+n-\lambda})c_o(m,\,n;\,q^{-\lambda})x^{-\lambda+2k+2l+m+n}, \tag{1.13}$$

where $\mathcal{P}_\lambda\subset(\mathbb{Z}_{\geqslant 0})^4$ denotes the finite set of points in the polyhedron defined by the set of inequalities

$$0\leqslant m\leqslant\lambda,\;\;0\leqslant n\leqslant\lambda-m,\;\;0\leqslant 2l\leqslant\lambda-m-n,\;\;0\leqslant k\leqslant\lambda-2l-m-n. \tag{1.14}$$

The even part of the series can be transformed by using Singh's formula or the q-analogue of Bailey's formula [2]. (See Propositions 4.2 and 4.3.)

Proposition 1.1. *We have the two bibasic representations with bases q and q^2:*

$$\sum_{k,l\geqslant 0} c_e(k,\,l;\,s\,|\,a,\,c\,|\,q)x^{2k+2l} = \sum_{k,l\geqslant 0} \frac{(qa^2/c^2;\,q^2)_k(q^{2l}s^2;\,q^2)_k}{(q^2;\,q^2)_k(q^{2l}q^3s^2/a^2c^2;\,q^2)_k}(q^2x^2/a^2)^k \times \frac{(c^2/q;\,q)_l(s;\,q)_l(q^2s^2/a^4;\,q^2)_l}{(q;\,q)_l(qs/a^2;\,q)_l(q^3s^2/a^2c^2;\,q^2)_l}(q^2x^2/c^2)^l$$

$$= \sum_{k,l\geqslant 0} \frac{(qa^2/c^2;\,q^2)_k(q^3s/c^2;\,q^2)_k(q^2s^2/c^4;\,q^2)_k}{(q^2;\,q^2)_k(qs/c^2;\,q^2)_k(q^3s^2/a^2c^2;\,q^2)_k}(q^2x^2/a^2)^k \times \frac{(c^2/q;\,q)_l(s;\,q)_{2k+l}}{(q;\,q)_l(q^2s/c^2;\,q)_{2k+l}}(q^2x^2/c^2)^l. \tag{1.15}$$

As an application, we present an explicit formula for Koornwinder polynomials [4] with one row diagram. It is derived from Theorem 1.2 and Proposition 1.1 by using the kernel function which intertwines the action of the Koornwinder operators of type BC_n and BC_1 [3] (see Section 5 below). Let $n \in \mathbb{Z}_{>0}$ and $x = (x_1,\,\ldots,\,x_n)$ be a set of variables. Let $P_{(r)}(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t)$ be the Koornwinder polynomial with one row diagram (r) $(r \in \mathbb{Z}_{\geqslant 0})$. Set

$$g_r(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t) = \frac{(t;\,q)_r}{(q;\,q)_r}P_{(r)}(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t), \tag{1.16}$$

for simplicity of display.

Definition 1.2. Define the symmetric Laurent polynomial $G_r(x;\,q,\,t)$ by

$$\prod_{i=1}^{n} \frac{(tux_i;\,q)_\infty}{(ux_i;\,q)_\infty}\frac{(tu/x_i;\,q)_\infty}{(u/x_i;\,q)_\infty} = \sum_{r\geqslant 0} G_r(x;\,q,\,t)u^r. \tag{1.17}$$

Theorem 1.4. *We have*

$$g_r(x\,|\,a,\,-a,\,c,\,-c\,|\,q,\,t) = \sum_{\substack{k,l\geqslant 0\\ 2k+2l\leqslant r}} G_{r-2k-2l}(x;\,q,\,t) \times \frac{(qa^2/c^2;\,q^2)_k(q^{3-r}t^{1-n}/c^2;\,q^2)_k(q^{2-2r}t^{2-2n}/c^4;\,q^2)_k}{(q^2;\,q^2)_k(q^{1-r}t^{1-n}/c^2;\,q^2)_k(q^{3-2r}t^{2-2n}/a^2c^2;\,q^2)_k}(t^2/a^2)^k \times \frac{(c^2/qt;\,q)_l(q^{-r}t^{-n};\,q)_{2k+l}}{(q;\,q)_l(q^{2-r}t^{1-n}/c^2;\,q)_{2k+l}}\frac{1-q^{-r+2k+2l}t^{-n}}{1-q^{-r}t^{-n}}(t^2/c^2)^l, \tag{1.18}$$

$$g_r(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t) = \sum_{\substack{i,j\geqslant 0\\ i+j\leqslant r}} g_{r-i-j}(x\,|\,a,\,-a,\,c,\,-c\,|\,q,\,t)\frac{(-b/a;\,q)_i(q^{1-r}t^{1-n}/cd;\,q)_i}{(q;\,q)_i(-q^{1-r}t^{1-n}/ac;\,q)_i} \times \frac{(q^{1-r}t^{-n};\,q)_{i+j}(-q^{1-r}t^{1-n}/ac;\,q)_{i+j}(q^{1-2r}t^{2-2n}/a^2c^2;\,q)_{i+j}}{(q^{2-2r}t^{2-2n}/abcd;\,q)_{i+j}(q^{1/2-r}t^{1-n}/ac;\,q)_{i+j}(-q^{1/2-r}t^{1-n}/ac;\,q)_{i+j}}(t/b)^i \times \frac{(-d/c;\,q)_j(q^{1-r}t^{1-n}/ab;\,q)_j}{(q;\,q)_j(-q^{1-r}t^{1-n}/ac;\,q)_j}(t/d)^j. \tag{1.19}$$

Corollary 1.1. *By specializing the parameters in* (1.19)*, we recover Lassalle's formulas for the Macdonald polynomials of type B, C and D, thereby proving his Conjectures* 1, 3 *and* 4 *in* [5].

Note that for Macdonald polynomials of type B, C and D, it is convenient to use the following simplified version of the series $f(x;\,s)$ (see Propositions 4.4 and 4.5).

Proposition 1.2. *We have*

$$f(x;\,s\,|\,-a,\,b,\,-q^{1/2}a,\,q^{1/2}b\,|\,q)=x^{-\lambda}\sum_{l,m\geqslant 0}\frac{(a^2;\,q)_m(q^ls;\,q)_m}{(q;\,q)_m(q^{l+1}s/a^2;\,q)_m}(qx^2/a^2)^m \times\frac{(b/a;\,q^{1/2})_l(s/a^2;\,q^{1/2})_l(s;\,q)_l}{(q^{1/2};\,q^{1/2})_l(q^{1/2}s/ab;\,q^{1/2})_l(s/a^2;\,q)_l}(q^{1/2}x/b)^l, \tag{1.20}$$

$$f(x;\,s\,|\,-a,\,b,\,-q^{1/2}a,\,q^{1/2}a\,|\,q)=x^{-\lambda}\sum_{l,m\geqslant 0}\frac{(a^2;\,q)_m(q^ls;\,q)_m}{(q;\,q)_m(q^{l+1}s/a^2;\,q)_m}(qx^2/a^2)^m \times\frac{(b/a;\,q)_l(s^2/a^4;\,q)_l(s;\,q)_l}{(q;\,q)_l(qs^2/a^3b;\,q)_l(s/a^2;\,q)_l}(qx/b)^l. \tag{1.21}$$

The structure of this paper is as follows. In Section 2, we give a proof of Theorem 1.1 (see Corollary 2.1). In Section 3, Theorem 1.2 is proved in two steps, first for the special parameters $(a,\,b,\,c,\,d)=(a,\,-a,\,c,\,-c)$ (Section 3.1, Theorem 3.1), and next for the general parameters $(a,\,b,\,c,\,d)$ (Section 3.2, Theorem 3.2). In Section 4, we prove Propositions 1.1 and 1.2 using some bibasic transformation formulas. Section 5 is devoted to the proof of Theorem 1.4. In Sections 5.2 and 5.3 the explicit formulas are derived for the Macdonald polynomials with one row for type C_n, B_n and D_n, thereby proving Lassalle's conjectures [5, p. 8, Conjecture 1, p. 10, Conjecture 3, p. 11, Conjecture 4]. In Appendix, some basic facts are recalled concerning the kernel function associated with Koornwinder's difference operator [3]. In Section 6.3, Theorem 6.1, a reproduction formula for the Koornwinder polynomials is given in terms of the kernel function. Some notation for the Macdonald polynomials of type B_n, C_n, and D_n are briefly given in Sections 6.4 and 6.5. In Section 6.7, we present a conjecture about the Macdonald polynomial of B_2.

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2. Proof of Theorem 1.1

For simplicity of display, we introduce a notation.

Definition 2.1. Set

$$\Psi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q)=\frac{(ax;\,q)_\infty}{(qx/a;\,q)_\infty}\sum_{n\geqslant 0}\frac{(qs^2/a^2;\,q)_n}{(q;\,q)_n}(ax/s)^n \times{}_6\phi_5\left[\begin{matrix}q^{-n},\,q^{n+1}s^2/a^2,\,s,\,qs/ab,\,qs/ac,\,qs/ad\\ q^2s^2/abcd,\,q^{1/2}s/a,\,-q^{1/2}s/a,\,qs/a,\,-qs/a\end{matrix};\,q,\,q\right]. \tag{2.1}$$

Theorem 2.1. *We have*

$$
{}_4\phi_3\left[\begin{matrix} q^{-m},\, abcdq^{m-1},\, ax,\, a/x \\ ab,\, ac,\, ad \end{matrix};\, q,\, q\right]
= a^m x^{-m}\frac{(abcdq^{m-1};\, q)_m}{(ab,\, ac,\, ad;\, q)_m}\Psi(x;\, q^{-m}\,|\,a,\, b,\, c,\, d\,|\,q). \quad (2.2)
$$

This gives an explicit formula for the infinite series $f(x;\, s)$ for generic $s \in \mathbb{C}$.

Corollary 2.1. *For generic* $s \in \mathbb{C}$, $f(x;\, s) = x^{-\lambda}\Psi(x;\, s\,|\,a,\, b,\, c,\, d\,|\,q)$ *satisfies* (1.7). *Hence Theorem 1.1 holds.*

Proof. The coefficients c_n's in (1.6) are clearly rational functions in s. After clearing the denominator $(1-x^2)(1-qx^2)(1-q^{-1}x^2)$ in (1.7), we have a set of linear relations for c_n's with coefficients being polynomials in s. Hence by substituting the c_n's (arising from $x^{-\lambda}\Psi(x;\, s\,|\,a,\, b,\, c,\, d\,|\,q)$) to these linear relations and clearing the denominators, we have a set of polynomials equations in s. Theorem 2.1 and (1.3) mean that these polynomials are zero for infinitely many points $s = q^{-n}$ $(n = 0, 1, 2, \ldots)$. Hence all of such polynomials are identically zero, indicating that (1.7) holds for generic $s \in \mathbb{C}$. □

Proof of Theorem 2.1. We need to expand the ${}_4\phi_3$ series in the form of x^{-m} times a power series in x. Therefore our starting point should be

$$
\text{LHS of (2.2)} = \sum_{k=0}^{m}\frac{(q^{-m},\, abcdq^{m-1},\, ax,\, a/x;\, q)_{m-k}}{(q,\, ab,\, ac,\, ad;\, q)_{m-k}}q^{m-k}. \quad (2.3)
$$

By using the q-binomial formula [2, p.7, (1.3.2)] we have

$$
\begin{aligned}
&(ax;\, q)_{m-k}(a/x;\, q)_{m-k} \\
&\quad = \frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty}\frac{(q^{-m+k+1}x/a;\, q)_\infty}{(q^{m-k}ax;\, q)_\infty}(-a/x)^{m-k}q^{(m-k)(m-k-1)/2} \\
&= \frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty}(-a/x)^{m-k}q^{(m-k)(m-k-1)/2}\sum_{l\geqslant 0}\frac{(q^{-2m+2k+1}/a^2;\, q)_l}{(q;\, q)_l}(q^{m-k}ax)^l.
\end{aligned} \quad (2.4)
$$

Then simplifying the factors we have

$$
\begin{aligned}
\text{RHS of (2.3)} &= a^m x^{-m}\frac{(abcdq^{m-1};\, q)_m}{(ab,\, ac,\, ad;\, q)_m}\frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty} \\
&\times\sum_{k\geqslant 0}\sum_{l\geqslant 0}\frac{(q^{-m},\, q^{-m+1}/ab,\, q^{-m+1}/ac,\, q^{-m+1}/ad;\, q)_k}{(q,\, q^{-2m+2}/abcd;\, q)_k} \\
&\times\frac{(q^{-2m+2k+1}/a^2;\, q)_l}{(q;\, q)_l}(-1)^k a^{k+l}x^{k+l}q^{(m-k)(k+l)+\frac{k(k+1)}{2}}
\end{aligned}
$$

$$= a^m x^{-m} \frac{(abcdq^{m-1};\, q)_m}{(ab,\, ac,\, ad;\, q)_m} \frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty} \sum_{n\geqslant 0} \frac{(q^{-2m+1}/a^2;\, q)_n}{(q;\, q)_n} (q^m ax)^n$$
$$\times \sum_{k\geqslant 0} \frac{(q^{-m},\, q^{-m+1}/ab,\, q^{-m+1}/ac,\, q^{-m+1}/ad;\, q)_k}{(q,\, q^{-2m+2}/abcd;\, q)_k} \frac{(q^{-n},\, q^{-2m+n+1}/a^2;\, q)_k}{(q^{-2m+1}/a^2;\, q)_{2k}} q^k$$
$$= \text{RHS of (2.2).} \tag{2.5}$$

□

3. Proof of Theorem 1.2

We embark on the proof of Theorem 1.2. For clarity of display we need another notation.

Definition 3.1. Set

$$\Phi(x;\, s\,|\, a,\, b,\, c,\, d\,|\, q) = \sum_{k,l,m,n\geqslant 0} c_e(k,\, l;\, q^{m+n}s\,|\, a,\, c\,|\, q) c_o(m,\, n;\, s\,|\, a,\, b,\, c,\, d\,|\, q) x^{2k+2l+m+n}. \tag{3.1}$$

In view of Corollary 2.1, we only need to show that $\Phi(x;\, s\,|\, a,\, b,\, c,\, d\,|\, q) = \Psi(x;\, s\,|\, a,\, b,\, c,\, d\,|\, q)$. We shall divide our proof in two steps. First, we will consider the special case $b = -a$, $d = -c$ in Section 3.1. Then the the general case will be treated in Section 3.2.

We remark that the special case $b = -a$, $d = -c$ (Theorem 3.1 below) constitute the essential part of the proof of Theorem 1.2. Based on Theorem 3.1, the general case can be treated easily (Theorem 3.2 below).

3.1. Case $b = -a$, $d = -c$. For the sake of clarity we first write down the expressions for $\Phi(x;\, s\,|\, a,\, -a,\, c,\, -c\,|\, q)$ and $\Psi(x;\, s\,|\, a,\, -a,\, c,\, -c\,|\, q)$ explicitly

$$\Phi(x;\, s\,|\, a,\, -a,\, c,\, -c\,|\, q) = \sum_{k,l\geqslant 0} c_e(k,\, l;\, s) x^{2k+2l}, \tag{3.2}$$

$$\Psi(x;\, s\,|\, a,\, -a,\, c,\, -c\,|\, q) = \frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty} \sum_{n\geqslant 0} \frac{(qs^2/a^2;\, q)_n}{(q;\, q)_n} (ax/s)^n$$
$$\times\, {}_6\phi_5 \left[\begin{matrix} q^{-n},\, q^{n+1}s^2/a^2,\, s,\, -qs/a^2,\, qs/ac,\, -qs/ac \\ q^2s^2/a^2c^2,\, q^{1/2}s/a,\, -q^{1/2}s/a,\, qs/a,\, -qs/a \end{matrix} ;\, q,\, q \right]. \tag{3.3}$$

Theorem 3.1. *We have*

$$\Phi(x;\, s\,|\, a,\, -a,\, c,\, -c\,|\, q) = \Psi(x;\, s\,|\, a,\, -a,\, c,\, -c\,|\, q). \tag{3.4}$$

Recall Verma's q-extension of the Field and Wimp expansion [2, p. 76, (3.7.9)]

$${}_{r+t}\phi_{s+u} \left[\begin{matrix} a_R,\, c_T \\ b_S,\, d_U \end{matrix} ;\, q,\, xw \right] = \sum_{j=0}^{\infty} \frac{(c_T,\, e_K;\, q)_j}{(q,\, d_U,\, \gamma q^j;\, q)_j} x^j [(-1)^j q^{\binom{j}{2}}]^{u+3-t-k}$$
$$\times\, {}_{t+k}\phi_{u+1} \left[\begin{matrix} c_T q^j,\, e_K q^j \\ \gamma q^{2j+1},\, d_U q^j \end{matrix} ;\, q,\, xq^{j(u+2-t-k)} \right] {}_{r+2}\phi_{s+k} \left[\begin{matrix} q^{-j},\, \gamma q^j,\, a_R \\ b_S,\, e_K \end{matrix} ;\, q,\, wq \right]. \tag{3.5}$$

Here we have used the contracted notation a_R for $a_1,\, \ldots,\, a_r$, etc.

Set the parameters in (3.5) as

$$r=2,\quad s=2,\quad t=4,\quad u=3,\quad k=1, \tag{3.6}$$

$$w=1,\quad x=q, \tag{3.7}$$

$$\begin{gathered} a_R=(qs/ac,\ -qs/ac),\quad b_S=(q^{1/2}s/a,\ -q^{1/2}s/a),\\ c_T=(q^{-n},\ q^{n+1}s^2/a^2,\ s,\ -qs/a^2),\quad d_U=(qs/a,\ -qs/a,\ q^2s^2/a^2c^2),\\ e_K=q^2s^2/a^2c^2,\quad \gamma=s^2/a^2. \end{gathered} \tag{3.8}$$

Then the choice of the parameters (3.6) means that we have an expansion of the form ${}_6\phi_5=\sum {}_5\phi_4\cdot {}_4\phi_3$. Note, however, that this ${}_5\phi_4$ series degenerates to a ${}_4\phi_3$ series from the conditions (3.8).

Lemma 3.1. *We have*

$$\begin{aligned} &{}_6\phi_5\left[{q^{-n},\ q^{n+1}s^2/a^2,\ s,\ -qs/a^2,\ qs/ac,\ -qs/ac \atop q^2s^2/a^2c^2,\ q^{1/2}s/a,\ -q^{1/2}s/a,\ qs/a,\ -qs/a};q,\ q\right]\\ &\quad=\sum_{j\geqslant 0}\frac{(q^{-n},\ q^{n+1}s^2/a^2,\ s,\ -qs/a^2,\ q^2s^2/a^2c^2;\ q)_j}{(q,\ qs/a,\ -qs/a,\ q^2s^2/a^2c^2,\ q^js^2/a^2;\ q)_j}(-1)^jq^{j+\binom{j}{2}}\\ &\qquad\times{}_4\phi_3\left[{q^{-n+j},\ q^{j+n+1}s^2/a^2,\ q^js,\ -q^{j+1}s/a^2 \atop q^{2j+1}s^2/a^2,\ q^{j+1}s/a,\ -q^{j+1}s/a};q,\ q\right]\\ &\qquad\times{}_4\phi_3\left[{q^{-j},\ q^js^2/a^2,\ qs/ac,\ -qs/ac \atop q^{1/2}s/a,\ -q^{1/2}s/a,\ q^2s^2/a^2c^2};q,\ q\right]. \end{aligned} \tag{3.9}$$

We need to transform the two ${}_4\phi_3$ series on RHS of (3.9). For the treatment of the last factor, we recall Andrews' terminating q-analogue of Watson's ${}_3F_2$ sum [2, p. 237, (II.17)]. Namely, we have

$${}_4\phi_3\left[{q^{-n},\ aq^n,\ c,\ -c \atop (aq)^{1/2},\ -(aq)^{1/2},\ c^2};q,\ q\right]=\begin{cases}0, & \text{if } n \text{ is odd},\\ \dfrac{c^n(q,\ aq/c^2;\ q^2)_{n/2}}{(aq,\ c^2q;\ q^2)_{n/2}}, & \text{if } n \text{ is even}.\end{cases} \tag{3.10}$$

Lemma 3.2. *From Andrews' formula, we have*

$$\begin{aligned} &{}_4\phi_3\left[{q^{-j},\ q^js^2/a^2,\ qs/ac,\ -qs/ac \atop q^{1/2}s/a,\ -q^{1/2}s/a,\ q^2s^2/a^2c^2};q,\ q\right]\\ &\quad=\begin{cases}0, & \textit{if } j \textit{ is odd},\\ \left(\dfrac{qs}{ac}\right)^j\dfrac{(q,\ c^2/q;\ q^2)_{j/2}}{(qs^2/a^2,\ q^3s^2/a^2c^2;\ q^2)_{j/2}}, & \textit{if } j \textit{ is even}.\end{cases} \end{aligned} \tag{3.11}$$

Hence the support of the sum in (3.9) is restricted to $j=0,\ 2,\ 4,\ \ldots$.

For the sake of simplicity, we shall change our notation in (3.9) as $j\to 2j$ and take summation over $j=0,\ 1,\ 2,\ \ldots$.

Next, we turn to the other ${}_4\phi_3$ series. Recall Singh's quadratic transformation [2, p. 89, (3.10.13)], that is,

$$ {}_4\phi_3\left[\begin{matrix} a^2,\, b^2,\, c,\, d \\ abq^{1/2},\, -abq^{1/2},\, -cd \end{matrix};\, q,\, q\right] = {}_4\phi_3\left[\begin{matrix} a^2,\, b^2,\, c^2,\, d^2 \\ a^2b^2q,\, -cd,\, -cdq \end{matrix};\, q^2,\, q^2\right], \tag{3.12} $$

provided the series terminates.

Lemma 3.3. *For $n,\, j \in \mathbb{Z}_{\geqslant 0}$ satisfying $2j \leqslant n$, we have the identity between the two terminating ${}_4\phi_3$ series*

$$ \begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n+2j},\, q^{2j+n+1}s^2/a^2,\, q^{2j}s,\, -q^{2j+1}s/a^2 \\ q^{4j+1}s^2/a^2,\, q^{2j+1}s/a,\, -q^{2j+1}s/a \end{matrix};\, q,\, q\right] \\ &= \frac{(q/a^2;\, q)_{n-2j}}{(q^{4j+1}s^2/a^2;\, q)_{n-2j}}(q^{2j}s)^{n-2j}\, {}_4\phi_3\left[\begin{matrix} q^{-n+2j},\, q^{-n+2j+1},\, q^{4j}s^2,\, a^2 \\ q^{4j+2}s^2/a^2,\, q^{-n+2j}a^2,\, q^{-n+2j+1}a^2 \end{matrix};\, q^2,\, q^2\right]. \end{aligned} \tag{3.13} $$

Proof of Lemma 3.3. It is enough to show (3.13) for the infinite discrete set of values $q^{2j}s = q^{-k}$, $(k = 0,\, 1,\, 2,\, \ldots)$. Under this condition, we can apply Singh's formula [2, p. 89, (3.10.13)], and we have

$$ \begin{aligned} \text{LHS of (3.13)} &= {}_4\phi_3\left[\begin{matrix} q^{-n+2j},\, q^{2j+n+1}s^2/a^2,\, q^{4j}s^2,\, q^{4j+2}s^2/a^4 \\ q^{4j+2}s^2/a^2,\, q^{4j+1}s^2/a^2,\, q^{4j+2}s^2/a^2 \end{matrix};\, q^2,\, q^2\right] \\ &= \frac{(q^{2j+n+1}s^2/a^2,\, q^{2j+n+2}s^2/a^2;\, q^2)_k}{(q^{4j+1}s^2/a^2,\, q^{4j+2}s^2/a^2;\, q^2)_k}(q^{-n+2j})^k \\ &\quad \times {}_4\phi_3\left[\begin{matrix} q^{-n+2j},\, q^{-n+2j+1},\, q^{4j}s^2,\, a^2 \\ q^{4j+2}s^2/a^2,\, q^{-n+2j}a^2,\, q^{-n+2j+1}a^2 \end{matrix};\, q^2,\, q^2\right], \end{aligned} \tag{3.14} $$

where we used Sears' transformation [2, p.41, (2.10.4)] in the second line. By noting $q^{2j}s = q^{-k}$, the factor in front of the ${}_4\phi_3$ series can be calculated as

$$ (q^{-n+2j})^k\frac{(q^{2j+n+1}s^2/a^2;\, q)_{2k}}{(q^{4j+1}s^2/a^2;\, q)_{2k}} = (q^{2j}s)^{n-2j}\frac{(q/a^2;\, q)_{n-2j}}{(q^{4j+1}s^2/a^2;\, q)_{n-2j}}. \tag{3.15} $$

□

Summarizing the results in Lemmas 3.1, 3.2 and 3.3, we arrive at the following intermediate expression.

Lemma 3.4. *We have*

$$ \text{RHS of (3.4)} = \frac{(ax;\, q)_\infty}{(qx/a;\, q)_\infty}\sum_{n\geqslant 0}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor \frac{n-2j}{2}\rfloor} A(n,\, j,\, m), \tag{3.16} $$

where

$$ \begin{aligned} A(n,\, j,\, m) = {}& \frac{(qs^2/a^2;\, q)_n}{(q;\, q)_n}(ax/s)^n\frac{(q^{-n},\, q^{n+1}s^2/a^2,\, s,\, -qs/a^2;\, q)_{2j}}{(q,\, qs/a,\, -qs/a,\, q^{2j}s^2/a^2;\, q)_{2j}}q^{j(2j+1)} \\ &\times \frac{(q/a^2;\, q)_{n-2j}}{(q^{4j+1}s^2/a^2;\, q)_{n-2j}}(q^{2j}s)^{n-2j}\frac{(q,\, c^2/q;\, q^2)_j}{(qs^2/a^2,\, q^3s^2/a^2c^2;\, q^2)_j}\left(\frac{qs}{ac}\right)^{2j} \\ &\times \frac{(q^{-n+2j},\, q^{-n+2j+1},\, q^{4j}s^2,\, a^2;\, q^2)_{2m}.}{(q^2,\, q^{4j+2}s^2/a^2,\, q^{-n+2j}a^2,\, q^{-n+2j+1}a^2;\, q^2)_{2m}}q^{2m}. \end{aligned} \tag{3.17} $$

Hence we can change the order of the summation as

$$\sum_{n\geqslant 0}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{m=0}^{\lfloor\frac{n-2j}{2}\rfloor} A(n,\,j,\,m)=\sum_{l\geqslant 0}\sum_{j\geqslant 0}\sum_{m\geqslant 0} A(l+2j+2m,\,j,\,m). \tag{3.18}$$

Lemma 3.5. *We have*

$$\frac{A(l+2j+2m,\,j,\,m)}{A(2j+2m,\,j,\,m)}=\frac{(q/a^2;\,q)_l}{(q;\,q)_l}(ax)^l, \tag{3.19}$$

$$A(2j+2m,\,j,\,m)=c_e(m,\,j;\,s)x^{2m+2j}. \tag{3.20}$$

Proof. Straightforward calculation. □

Now we are ready to present the final step.

Proof of Theorem 3.1. From Lemmas 3.4, 3.5 and the q-binomial formula, we have

$$\begin{aligned}\text{RHS of (3.4)}&=\frac{(ax;\,q)_\infty}{(qx/a;\,q)_\infty}\sum_{l\geqslant 0}\sum_{j\geqslant 0}\sum_{m\geqslant 0}A(l+2j+2m,\,j,\,m)\\ &=\sum_{j\geqslant 0}\sum_{m\geqslant 0}A(2j+2m,\,j,\,m)=\Phi(x;\,s\,|\,a,\,-a,\,c,\,-c\,|\,q)=\text{LHS of (3.4)}.\end{aligned} \tag{3.21}$$

□

3.2. General Case.

Based on Theorem 3.1, we treat the general case.

Theorem 3.2. *We have*

$$\Psi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q)=\Phi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q), \tag{3.22}$$

which implies Theorem 1.2.

Proof. We use Verma's formula once again. In (3.5) set

$$r=2,\quad s=1,\quad t=4,\quad u=4,\quad k=2, \tag{3.23}$$

$$w=1,\quad x=q, \tag{3.24}$$

$$\begin{gathered}a_R=(qs/ab,\,qs/ad),\quad b_S=q^2s^2/abcd,\quad c_T=(q^{-n},\,q^{n+1}s^2/a^2,\,s,\,qs/ac),\\ d_U=(q^{1/2}s/a,\,-q^{1/2}s/a,\,qs/a,\,-qs/a),\quad e_K=(-qs/a^2,\,-qs/ac),\\ \gamma=qs^2/a^2c^2.\end{gathered} \tag{3.25}$$

The choice of the parameters (3.23) means that we have the expansion of the form ${}_6\phi_5=\sum{}_6\phi_5\cdot{}_4\phi_3$. Note that the parameters in (3.24) and (3.25) are chosen in such a way that the structure of the resulting ${}_6\phi_5$ possesses the same structure as the one in $\Psi(x;\,a\,|\,a,\,-a,\,c,\,-c\,|\,q)$.

Lemma 3.6. *The following equality holds*

$$\Psi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q) = \frac{(ax;\,q)_\infty}{(qx/a;\,q)_\infty}\sum_{n\geqslant 0}\frac{(qs^2/a^2;\,q)_n}{(q;\,q)_n}(ax/s)^n$$
$$\times\sum_{j\geqslant 0}\frac{(q^{-n},\,q^{n+1}s^2/a^2,\,s,\,qs/ac,\,-qs/a^2,\,-qs/ac;\,q)_j}{(q,\,q^{1/2}s/a,\,-q^{1/2}s/a,\,qs/a,\,-qs/a,\,q^{j+1}s^2/a^2c^2;\,q)_j}(-1)^j q^{j+\binom{j}{2}}$$
$$\times\,{}_6\phi_5\left[\begin{matrix}q^{-n+j},\,q^{n+j+1}s^2/a^2,\,q^js,\,q^{j+1}s/ac,\,-q^{j+1}s/a^2,\,-q^{j+1}s/ac\\ q^{2j+2}s^2/a^2c^2,\,q^{1/2+j}s/a,\,-q^{1/2+j}s/a,\,q^{j+1}s/a,\,-q^{j+1}s/a\end{matrix};\,q,\,q\right]$$
$$\times\,{}_4\phi_3\left[\begin{matrix}q^{-j},\,q^{j+1}s^2/a^2c^2,\,qs/ab,\,qs/ad\\ q^2s^2/abcd,\,-qs/a^2,\,-qs/ac\end{matrix};\,q,\,q\right]. \tag{3.26}$$

Set $n = m+j$, change the order of the summation as $\sum_{n=0}^{\infty}\sum_{j=0}^{n} = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}$, and apply Theorem 3.1 to the RHS of (3.26).

Lemma 3.7. *We have*

$$\Psi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q) = \sum_{j\geqslant 0}\sum_{k,l\geqslant 0} c_e(k,\,l;\,q^js)x^{2k+2l}\frac{(qs^2/a^2;\,q)_j}{(q;\,q)_j}(ax/s)^j$$
$$\times\frac{(q^{-j},\,q^{j+1}s^2/a^2,\,s,\,qs/ac,\,-qs/a^2,\,-qs/ac;\,q)_j}{(q,\,q^{1/2}s/a,\,-q^{1/2}s/a,\,qs/a,\,-qs/a,\,q^{j+1}s^2/a^2c^2;\,q)_j}(-1)^j q^{j+\binom{j}{2}}$$
$$\times\,{}_4\phi_3\left[\begin{matrix}q^{-j},\,q^{j+1}s^2/a^2c^2,\,qs/ab,\,qs/ad\\ q^2s^2/abcd,\,-qs/a^2,\,-qs/ac\end{matrix};\,q,\,q\right]. \tag{3.27}$$

Applying the Sears' transformation and simplifying the factors, we get

$$\text{RHS of (3.27)} = \sum_{j\geqslant 0}\sum_{k,l\geqslant 0} c_e(k,\,l;\,q^js)x^{2k+2l}$$
$$\times\frac{(-d/c,\,qs/ab,\,s,\,qs^2/a^2c^2;\,q)_j}{(q,\,q^2s^2/abcd,\,q^{1/2}s/ac,\,-q^{1/2}s/ac;\,q)_j}(qx/d)^j$$
$$\times\,{}_4\phi_3\left[\begin{matrix}q^{-j},\,-b/a,\,qs/cd,\,-q^{-j}ac/s\\ -qs/ac,\,-q^{-j+1}c/d,\,q^{-j}ab/s\end{matrix};\,q,\,q\right]. \tag{3.28}$$

Lemma 3.8. *We have*

$$\sum_{m,n\geqslant 0} c_o(m,\,n;\,s)x^{m+n} = \sum_{l\geqslant 0}x^l\sum_{m=0}^{l}c_o(m,\,l-m;\,s)$$
$$= \sum_{l\geqslant 0}\frac{(-d/c,\,qs/ab,\,s,\,qs^2/a^2c^2;\,q)_l}{(q,\,q^2s^2/abcd,\,q^{1/2}s/ac,\,-q^{1/2}s/ac;\,q)_l}(qx/d)^l$$
$$\times\,{}_4\phi_3\left[\begin{matrix}q^{-l},\,-q^{-l}ac/s,\,-b/a,\,qs/cd\\ -q^{-l+1}c/d,\,q^{-l}ab/s,\,-qs/ac\end{matrix};\,q,\,q\right]. \tag{3.29}$$

Proof. Straightforward calculation. □

Summarizing these, we get

$$\text{RHS of (3.28)} = \sum_{m,n\geqslant 0}\sum_{k,l\geqslant 0} c_e(k,\, l;\, q^{m+n}s)c_o(m,\, n;\, s)x^{2k+2l+m+n} = \Phi(x;\, s\,|\,a,\, b,\, c,\, d\,|\,q), \tag{3.30}$$

thereby completing the proof of Theorem 3.2. □

4. Bibasic Transformations for $c_e(k,\, l;\, s)$ and $c_o(k,\, l;\, s)$

In this section, transformation formulas for the series $\sum_{l=0}^{k} c_e(k-l,\, l;\, s)$ and $\sum_{l=0}^{k} c_o(k-l,\, l;\, s)$ are given.

4.1. Transformation by Watson's formula.

Proposition 4.1. *We have*

$$\sum_{l=0}^{k} c_e(k-l,\, l;\, s) = \frac{(a^2c^2/q,\, s^2;\, q^2)_k}{(q^2,\, q^3s^2/a^2c^2;\, q^2)_k}(q^3/a^2c^2)^k\, {}_4\phi_3\left[\begin{matrix} a^2/q,\, c^2/q,\, q^{2k}s^2,\, q^{-2k} \\ -s,\, -qs,\, a^2c^2/q \end{matrix};\, q^2,\, q^2\right]. \tag{4.1}$$

Remark 4.1. This is manifestly symmetric in the exchange of a and c.

Proof. From the definition (1.9), we have

$$\sum_{l=0}^{k} \frac{c_e(k-l,\, l;\, s)}{c_e(k,\, 0;\, s)} = {}_8\phi_7\left[\begin{matrix} s^2/a^2,\, q^2s/a,\, -q^2s/a,\, q^{2k}s^2,\, c^2/q,\, -qs/a^2,\, -q^2s/a^2,\, q^{-2k} \\ s/a,\, -s/a,\, q^{-2k+2}/a^2,\, q^3s^2/a^2c^2,\, -qs,\, -s,\, q^{2k+2}s^2/a^2 \end{matrix};\, q^2,\, q^2/c^2\right]. \tag{4.2}$$

Then we apply Watson's transformation formula [2, p. 35, (2.5.1)] to the ${}_8\phi_7$ series. Simplifying the factors, we get (4.1). □

4.2. Bibasic transformation by Singh's formula.

Proposition 4.2. *We have*

$$\sum_{k,l\geqslant 0} c_e(k,\, l;\, s)x^{2k+2l} = \sum_{k,l\geqslant 0} \frac{(qa^2/c^2,\, q^{2l}s^2;\, q^2)_k}{(q^2,\, q^{2l}q^3s^2/a^2c^2;\, q^2)_k}(q^2x^2/a^2)^k \times \frac{(c^2/q,\, s;\, q)_l(q^2s^2/a^4;\, q^2)_l}{(q,\, qs/a^2;\, q)_l(q^3s^2/a^2c^2;\, q^2)_l}(q^2x^2/c^2)^l. \tag{4.3}$$

Proof. Applying Singh's formula and Sears' transformation to the ${}_4\phi_3$ series in (4.1), we have

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}a^2/q,\, c^2/q,\, q^{2k}s^2,\, q^{-2k}\\ -s,\, -qs,\, a^2c^2/q\end{matrix};q^2,\, q^2\right]={}_4\phi_3\left[\begin{matrix}a^2/q,\, c^2/q,\, q^ks,\, q^{-k}\\ q^{-1/2}ac,\, -q^{-1/2}ac,\, -s\end{matrix};q,\, q\right] \quad (4.4)\\ &=\frac{(qa^2/c^2;\, q^2)_k}{(a^2c^2/q;\, q^2)_k}(c^2/q)^k{}_4\phi_3\left[\begin{matrix}q^{-k}-q^{-k},\, c^2/q,\, -qs/a^2\\ -s,\, q^{-k+1/2}c/a,\, -q^{-k+1/2}c/a\end{matrix};q,\, q\right].\end{aligned}$$

Simplifying the factors, we get (4.3). □

4.3. Bibasic transformation by q-analogue of Bailey's formula

Proposition 4.3. *We have*

$$\begin{aligned}&\sum_{k,l\geqslant 0}c_e(k,\, l;\, s)x^{2k+2l}\\ &=\sum_{k,l\geqslant 0}\frac{(qa^2/c^2;\, q^2)_k(s;\, q^2)_k(qs;\, q^2)_k(q^2s^2/c^4;\, q^2)_k}{(q^2;\, q^2)_k(qs/c^2;\, q^2)_k(q^2s/c^2;\, q^2)_k(q^3s^2/a^2c^2;\, q^2)_k}(q^2x^2/a^2)^k\\ &\qquad\times\frac{(c^2/q;\, q)_l(q^{2k}s;\, q)_l}{(q;\, q)_l(q^{2k}q^2s/c^2;\, q)_l}(q^2x^2/c^2)^l\\ &=\sum_{k,l\geqslant 0}\frac{(qa^2/c^2;\, q^2)_k(q^3s/c^2;\, q^2)_k(q^2s^2/c^4;\, q^2)_k}{(q^2;\, q^2)_k(qs/c^2;\, q^2)_k(q^3s^2/a^2c^2;\, q^2)_k}(q^2x^2/a^2)^k\\ &\qquad\times\frac{(c^2/q;\, q)_l(s;\, q)_{2k+l}}{(q;\, q)_l(q^2s/c^2;\, q)_{2k+l}}(q^2x^2/c^2)^l. \quad (4.5)\end{aligned}$$

Proof. The coefficient of x^{2l} on RHS of (4.5) reads

$$\begin{aligned}&\frac{(c^2/q;\, q)_l(s;\, q)_l}{(q;\, q)_l(q^2s/c^2;\, q)_l}(q^2/c^2)^l\\ &\quad\times\sum_{k=0}^{l}\frac{(q^2s^2/c^4,\, q^3s/c^2,\, qa^2/c^2;\, q^2)_k}{(q^2,\, qs/c^2,\, q^3s^2/a^2c^2;\, q^2)_k}\frac{(q^ls,\, q^{-l};\, q)_k}{(q^{-l+2}/c^2,\, q^{l+2}s/c^2;\, q)_k}(q^2/a^2)^k. \quad (4.6)\end{aligned}$$

The second factor can be expressed in terms of the bibasic hypergeometric series as

$$\Phi\left[\begin{matrix}q^2s^2/c^4,\, q^3s/c^2,\, qa^2/c^2 : q^ls,\, q^{-l}\\ qs/c^2,\, q^3s^2/a^2c^2 : q^{-l+2}/c^2,\, q^{l+2}s/c^2\end{matrix};q^2,\, q;\, q^2/a^2\right], \quad (4.7)$$

where we used the notation for the bibasic hypergeometric series (see [2, p. 85, (3.9.1)])

$$\Phi\left[\begin{matrix}a_1,\, \dots,\, a_{r+1} : c_1,\, \dots,\, c_s\\ b_1,\, \dots,\, b_r : d_1,\, \dots,\, d_s\end{matrix};q,\, p;\, z\right]=\sum_{n=0}^{\infty}\frac{(a_1,\, \dots,\, a_{r+1};\, q)_n}{(q,\, b_1,\, \dots,\, b_r;\, q)_n}\frac{(c_1,\, \dots,\, c_s;\, p)_n}{(d_1,\, \dots,\, d_s;\, p)_n}z^n. \quad (4.8)$$

Recall the q-analogue of Bailey's transformation [2, p. 89, (3.10.14)]

$$\Phi\left[\begin{matrix}a^2,\, aq^2,\, -aq^2,\, b^2,\, c^2 : -aq/w,\, q^{-n}\\ a,\, -a,\, a^2q^2/b^2,\, a^2q^2/c^2 : w,\, -aq^{n+1}\end{matrix};\, q^2,\, q;\, \frac{awq^{n+1}}{b^2c^2}\right]$$
$$= \frac{(w/a,\, -aq;\, q)_n}{(w,\, -q;\, q)_n}\frac{(wq^{-n-1}/a,\, aq^{2-n}/w;\, q^2)_n}{(aq^{1-n}/w,\, wq^{-n}/a;\, q^2)_n}$$
$$\times\, {}_5\phi_4\left[\begin{matrix}aq,\, aq^2,\, a^2q^2/b^2c^2,\, a^2q^2/w^2,\, q^{-2n}\\ a^2q^2/b^2,\, a^2q^2/c^2,\, aq^{2-n}/w,\, aq^{3-n}/w\end{matrix};\, q^2,\, q^2\right]. \quad (4.9)$$

Note that setting $a = c^2$ and replacing the parameters as

$$(a,\, w,\, b^2) \to (-qs/c^2,\, q^{-l+2}/c^2,\, qa^2/c^2)$$

in (4.9) we have a bibasic transformation formula involving a ${}_4\phi_3$ series which applies to (4.7). The resulting ${}_4\phi_3$ has the structure

$$ {}_4\phi_3\left[\begin{matrix}-q^2s/a^2,\, -q^2s/c^2,\, q^{2l}s^2,\, q^{-2l}\\ -s,\, -qs,\, q^3s^2/a^2c^2\end{matrix};\, q^2,\, q^2\right], \quad (4.10)$$

which is transformed by Sears' formula into

$$ {}_4\phi_3\left[\begin{matrix}a^2/q,\, c^2/q,\, q^{2l}s^2,\, q^{-2l}\\ -s,\, -qs,\, a^2c^2/q\end{matrix};\, q^2,\, q^2\right], \quad (4.11)$$

up to a multiplicative factor given in terms of q-shifted factorials. Summarizing these, one finds that RHS of (4.5) is transformed to RHS of (4.1). □

4.4. Simplification of $\Phi(x;\, s\,|\,{-a},\, b,\, -q^{1/2}a,\, q^{1/2}b\,|\,q)$. One finds a simplification of the $\Phi(x; s)$ to a twofold series with bases $q^{1/2}$ and q when we specialize the parameters as $(a,\, b,\, c,\, d) \to (-a,\, b,\, -q^{1/2}a,\, q^{1/2}b)$.

Proposition 4.4. *We have*

$$\Phi(x;\, s\,|\,{-a},\, b,\, -q^{1/2}a,\, q^{1/2}b\,|\,q) = \sum_{l,m\geqslant 0}\frac{(a^2;\, q)_m(q^ls;\, q)_m}{(q;\, q)_m(q^{l+1}s/a^2;\, q)_m}(qx^2/a^2)^m$$
$$\times\, \frac{(b/a;\, q^{1/2})_l(s/a^2;\, q^{1/2})_l(s;\, q)_l}{(q^{1/2};\, q^{1/2})_l(q^{1/2}s/ab;\, q^{1/2})_l(s/a^2;\, q)_l}(q^{1/2}x/b)^l. \quad (4.12)$$

Proof. The ${}_4\phi_3$ series on the RHS of (3.29) written for the parameters $(-a,\, b,\, -c,\, d)$ is transformed by Sears' formula as

$$ {}_4\phi_3\left[\begin{matrix}q^{-l},\, -q^{-l}ac/s,\, b/a,\, -qs/cd\\ q^{-l+1}c/d,\, -q^{-l}ab/s,\, -qs/ac\end{matrix};\, q,\, q\right]$$
$$= \frac{(bd/ac,\, -qs/bc;\, q)_l}{(d/c,\, -qs/ac;\, q)_l}\,{}_4\phi_3\left[\begin{matrix}q^{-l},\, q^{-l-1}abcd/s^2,\, b/a,\, b/c\\ -q^{-l}ab/s,\, -q^{-l}bc/s,\, bd/ac\end{matrix};\, q,\, q\right].$$

Note that if $c = q^{1/2}a$, $d = q^{1/2}b$, Singh's transformation applies and we can simplify the ${}_4\phi_3$ series as

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-l},\, q^{-l}a^2b^2/s^2,\, b/a,\, q^{-1/2}b/a \\ -q^{-l}ab/s,\, -q^{-l+1/2}ab/s,\, b^2/a^2 \end{matrix}; q,\, q\right] \\
&\qquad = {}_4\phi_3\left[\begin{matrix} q^{-l/2},\, q^{-l/2}ab/s,\, b/a,\, q^{-1/2}b/a \\ b/a,\, -b/a,\, -q^{-l}ab/s \end{matrix}; q^{1/2},\, q^{1/2}\right] \\
&\qquad = {}_3\phi_2\left[\begin{matrix} q^{-l/2},\, q^{-l/2}ab/s,\, q^{-1/2}b/a \\ -b/a,\, -q^{-l}ab/s \end{matrix}; q^{1/2},\, q^{1/2}\right] \\
&\qquad = \frac{(-q^{1/2},\, q^{l/2}s/a^2;\, q^{1/2})_l}{(-b/a,\, -q^{l/2+1/2}s/ab;\, q^{1/2})_l},
\end{aligned} \tag{4.13}$$

where we used the q-Saalschütz sum [2, p. 813, (1.7.2)]. Hence we have

$$\begin{aligned}
&\sum_{m=0}^{l} c_o(m,\, l-m;\, s\,|\,-a,\, b,\, -q^{1/2}a,\, q^{1/2}b\,|\,q) \\
&\qquad = \frac{(b/a;\, q^{1/2})_l(s/a^2;\, q^{1/2})_l(s;\, q)_l}{(q^{1/2};\, q^{1/2})_l(q^{1/2}s/ab;\, q^{1/2})_l(s/a^2;\, q)_l}(q^{1/2}/b)^l.
\end{aligned} \tag{4.14}$$

From (4.3) or (4.5), we have

$$\sum_{k=0}^{m} c_e(k,\, m-k;\, q^l s\,|\,-a,\, b,\, -q^{1/2}a,\, q^{1/2}b\,|\,q) = \frac{(a^2;\, q)_m(q^l s;\, q)_m}{(q;\, q)_m(q^{l+1}s/a^2;\, q)_m}(q/a^2)^m. \tag{4.15}$$

□

4.5. Simplification of $\Phi(x;\, s\,|\,-a,\, b,\, -q^{1/2}a,\, q^{1/2}a\,|\,q)$. One finds another simplification of the $\Phi(x;\, s)$ to a twofold series with base q when we specialize the parameters as $(a,\, b,\, c,\, d) \to (-a,\, b,\, -q^{1/2}a,\, q^{1/2}a)$.

Proposition 4.5. *We have*

$$\begin{aligned}
\Phi(x;\, s\,|\,-a,\, b,\, -q^{1/2}a,\, q^{1/2}a\,|\,q) &= \sum_{l,m\geqslant 0} \frac{(a^2;\, q)_m(q^l s;\, q)_m}{(q;\, q)_m(q^{l+1}s/a^2;\, q)_m}(qx^2/a^2)^m \\
&\quad \times \frac{(b/a;\, q)_l(s^2/a^4;\, q)_l(s;\, q)_l}{(q;\, q)_l(qs^2/a^3b;\, q)_l(s/a^2;\, q)_l}(qx/b)^l.
\end{aligned} \tag{4.16}$$

Proof. Simple calculation using (1.10) and (4.3) (or (4.5)). □

5. Proof of Theorem 1.4

We apply our formulas for $f(x;\, s)$ for Koornwinder polynomials with one row diagram. We use the kernel function of type BC which intertwines the action of the Koornwinder operators of type BC_n and type BC_1. As for the notation and basic facts about the kernel function of type BC, we refer the reader to [3] and to Section 6.

Lemma 5.1. *Note that from Proposition 4.3, we have*

$$(1-x^2)\sum_{k,l\geqslant 0} c_e(k,\,l;\,s)x^{2k+2l} = \sum_{k,l\geqslant 0} c'_e(k,\,l;\,s\,|\,a,\,c\,|\,q)x^{2k+2l}, \tag{5.1}$$

where

$$c'_e(k,\,l;\,s\,|\,a,\,c\,|\,q) = \frac{(qa^2/c^2;\,q^2)_k(q^3s/c^2;\,q^2)_k(q^2s^2/c^4;\,q^2)_k}{(q^2;\,q^2)_k(qs/c^2;\,q^2)_k(q^3s^2/a^2c^2;\,q^2)_k}(q^2/a^2)^k \\ \times \frac{(c^2/q^2;\,q)_l(s/q;\,q)_{2k+l}}{(q;\,q)_l(q^2s/c^2;\,q)_{2k+l}}\frac{1-q^{2k+2l-1}s}{1-q^{-1}s}(q^2/c^2)^l. \tag{5.2}$$

Proof. Straightforward. □

5.1. Koornwinder polynomial $P_{(r)}(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t)$ with one row diagram. We move on to the proof of Theorem 1.4. Recall that n is a positive integer, $x=(x_1,\,\ldots,\,x_n)$ is a set of variables, and $P_{(r)}(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t)$ denotes the Koornwinder polynomial with one row diagram (r).

Proof of Theorem 1.4. We consider the following special case of Theorem 6.1 below

$$x=(x_1,\,\ldots,\,x_n)\quad (n\in\mathbb{Z}_{>0}), \tag{5.3}$$

$$y=(y_1)\quad (m=1), \tag{5.4}$$

$$\Pi(x;\,y) = y^{\beta n}\prod_{i=1}^{n}\frac{(q^{1/2}t^{1/2}yx_i;\,q)_\infty}{(q^{1/2}t^{-1/2}yx_i;\,q)_\infty}\frac{(q^{1/2}t^{1/2}y/x_i;\,q)_\infty}{(q^{1/2}t^{-1/2}y/x_i;\,q)_\infty}, \tag{5.5}$$

$$\lambda=(r)\quad (r\in\mathbb{Z}_{\geqslant 0}), \tag{5.6}$$

$$s=(s_1)=q^{1-r}t^{-n}, \tag{5.7}$$

$$V(y)=y^{-1}(1-y^2), \tag{5.8}$$

$$\widehat{f}(x;\,s) = y^{-r+1-n\beta}\widehat{\Phi}(x;\,s) \\ = y^{-r+1-n\beta}\sum_{k,l,i,j\geqslant 0}\widehat{c}_e(k,\,l;\,q^{i+j}s)\widehat{c}_o(i,\,j;\,s)x^{2k+2l+i+j}, \tag{5.9}$$

where

$$\widehat{c}_e(k,\,l;\,s) = c_e(k,\,l;\,s\,|\,\sqrt{q/t}\,a,\,\sqrt{q/t}\,c\,|\,q), \tag{5.10}$$

$$\widehat{c}_o(i,\,j;\,s) = c_o(i,\,j;\,s\,|\,\sqrt{q/t}\,a,\,\sqrt{q/t}\,b,\,\sqrt{q/t}\,c,\,\sqrt{q/t}\,d\,|\,q). \tag{5.11}$$

Note that we have

$$y^{-n\beta}\Pi(x;\,y) = \sum_{r\geqslant 0} G_r(x;\,q,\,t)(q/t)^{r/2}y^r, \tag{5.12}$$

where G_r is defined in Definition 1.2.

Then we find the constant term in y, which is proportional to $P_{(r)}(x\,|\,a,b,c,d\,|\,q,t)$, as

$$
\begin{aligned}
\big[\Pi(x;\,y)V(y)\widehat{f}(y;\,s)\big]_{1,y} &= \Bigg[\sum_{\theta\geqslant 0} G_\theta(x;\,q,\,t)(q/t)^{\theta/2}y^\theta y^{-r}(1-y^2)\widehat{\Phi}(y;\,s)\Bigg]_{1,y} \\
&= \sum_{k,l,i,j\geqslant 0} G_{r-2k-2l-i-j}(x;\,q,\,t)(q/t)^{(r-2k-2l-i-j)/2} \\
&\qquad \times \widehat{c}'_e(k,\,l;\,q^{1-r+i+j}t^{-n})\widehat{c}_o(i,\,j;\,q^{1-r}t^{-n}) \\
&= (q/t)^{r/2}\frac{(t;\,q)_r}{(q;\,q)_r}P_{(r)}(x\,|\,q,\,b,\,c,\,d\,|\,q,\,t) = (q/t)^{r/2}g_r(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t), \qquad (5.13)
\end{aligned}
$$

where $\widehat{c}'_e(k,\,l;\,s) = c'_e(k,\,l;\,s\,|\,\sqrt{q/t}\,a,\,\sqrt{q/t}\,c\,|\,q)$ (see (5.2) above in Lemma 5.1). Here we have used

$$
G_r(x;\,q,\,t) = \frac{(t;\,q)_r}{(q;\,q)_r}\sum_{i=1}^{n} x_i^n + \text{lower degree terms}. \qquad (5.14)
$$

Setting $b=-a$ and $d=-c$ in (5.13), we get

$$
\sum_{k,l\geqslant 0} G_{r-2k-2l}(x;\,q,\,t)(q/t)^{-k-l}\,\widehat{c}'_e(k,\,l;\,q^{1-r}t^{-n}) = g_r(x\,|\,a,\,-a,\,c,\,-c\,|\,q,\,t), \qquad (5.15)
$$

because $\widehat{c}_o(i,\,j;\,s)=\delta_{i,0}\delta_{j,0}$ in this case. Hence we obtain (1.18). Using this and (5.13) once again, we get

$$
\begin{aligned}
\sum_{i,j\geqslant 0} g_{r-i-j}(x\,|\,a,\,-a,\,c,\,-c\,|\,q,\,t)(q/t)^{-(i+j)/2}\,\widehat{c}_o(i,\,j;\,q^{1-r}t^{-n}) \\
= g_r(x\,|\,a,\,b,\,c,\,d\,|\,q,\,t). \qquad (5.16)
\end{aligned}
$$

This proves (1.19). □

5.2. Macdonald polynomial $P^{(C_n)}_{(r)}(x\,|\,b;\,q,\,t)$ of type C_n with one row diagram. Let $P^{(C_n)}_\lambda(x\,|\,b;\,q,\,t)$ be the Macdonald polynomial of type C_n (as for the notation, see Section 6.4). In view of (6.25), we need the series $\Phi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q)$ specialized for the case

$$
(-(qb/t)^{1/2},\,(qb/t)^{1/2},\,-q^{1/2}(qb/t)^{1/2},\,q^{1/2}(qb/t)^{1/2},\,q), \qquad (5.17)
$$

for the construction of the formula for $P^{(C_n)}_{(r)}(x\,|\,q,\,t)$.

Lemma 5.2. *From (4.3) or (4.5), we have*

$$
\begin{aligned}
(1-x^2)\Phi(x;\,s\,|-a,\,a,\,-q^{1/2}a,\,q^{1/2}a\,|\,q) &= (1-x^2)\sum_{j\geqslant 0}\frac{(a^2;\,q)_j(s;\,q)_j}{(q;\,q)_j(qs/a^2;\,q)_j}(qx^2/a^2)^j \\
&= \sum_{j\geqslant 0}\frac{(a^2/q;\,q)_j(s/q;\,q)_j}{(q;\,q)_j(qs/a^2;\,q)_j}\frac{1-q^{2j-1}s}{1-q^{-1}s}(qx^2/a^2)^j. \qquad (5.18)
\end{aligned}
$$

Theorem 5.1. *We have*

$$
\begin{aligned}
&P^{(C_n)}_{(r)}(x)\\
&=\frac{(q;\,q)_r}{(t;\,q)_r}\sum_{j=0}^{\lfloor r/2\rfloor} G_{r-2j}(x;\,q,\,t)\frac{(b/t;\,q)_j(t^{-n}q^{-r};\,q)_j}{(q;\,q)_j(t^{-n+1}q^{-r+1}/b;\,q)_j}\frac{1-t^{-n}q^{-r+2j}}{1-t^{-n}q^{-r}}(t^2/qb)^j,
\end{aligned}
$$

thereby proving Lassalle's conjecture for type C [5, p. 8, Conjecture 1] (see Section 6.6).

5.3. Macdonald polynomials $P^{(B_n)}_{(r)}(x\,|\,a;\,q,\,t)$ and $P^{(D_n)}_{(r)}(x\,|\,q,\,t)$ of types B_n and D_n with one row diagram. For the construction of the formula for $P^{(B_n)}_{(r)}(x\,|\,a;\,q,\,t)$, we need $\Phi(x;\,s\,|\,a,\,b,\,c,\,d\,|\,q)$ specialized for the parameters

$$
(-(q/t)^{1/2},\,a(q/t)^{1/2},\,-qt^{-1/2},\,qt^{-1/2};\,q,\,q/t). \tag{5.19}
$$

Lemma 5.3. *We have*

$$
\begin{aligned}
&(1-x^2)\Phi(x;\,s\,|\,-a,\,b,\,-q^{1/2}a,\,q^{1/2}a\,|\,q)\\
&=(1-x^2)\sum_{i,j\geqslant 0}\frac{(a^2;\,q)_j(s;\,q)_{i+j}}{(q;\,q)_j(qs/a^2;\,q)_{i+j}}\frac{(b/a;\,q)_i(s^2/a^4;\,q)_i(qs/a^2;\,q)_i}{(q;\,q)_i(qs^2/a^3b;\,q)_i(s/a^2;\,q)_i}(qx^2/a^2)^j(qx/b)^i\\
&=\sum_{i,j\geqslant 0}\frac{(a^2/q;\,q)_j(s/q;\,q)_{i+j}}{(q;\,q)_j(qs/a^2;\,q)_{i+j}}\frac{(b/a;\,q)_i(s^2/a^4;\,q)_i(qs/a^2;\,q)_i}{(q;\,q)_i(qs^2/a^3b;\,q)_i(s/a^2;\,q)_i}\\
&\qquad\times\frac{1-q^{i+2j-1}s}{1-q^{-1}s}(qx^2/a^2)^j(qx/b)^i.
\end{aligned}\tag{5.20}
$$

Theorem 5.2. *We have*

$$
\begin{aligned}
P^{(B_n)}_{(r)}(x\,|\,a;\,q,\,t)&=\frac{(q;\,q)_r}{(t;\,q)_r}\sum_{0\leqslant i+2j\leqslant r}G_{r-i-2j}(x;\,q,\,t)\\
&\times\frac{(a;\,q)_i(t^{-n+1}q^{-r+1};\,q)_i(t^{-n}q^{-r};\,q)_{i+j}(t^{-2n+2}q^{-2r};\,q)_i(1/t;\,q)_j}{(q;\,q)_i(t^{-n+1}q^{-r};\,q)_i(t^{-n+1}q^{-r+1};\,q)_{i+j}(t^{-2n+2}q^{-2r+1}/a;\,q)_i(q;\,q)_j}\\
&\times\frac{1-t^{-n}q^{-r+i+2j}}{1-t^{-n}q^{-r}}(t/a)^i(t^2/q)^j.
\end{aligned}\tag{5.21}
$$

By setting $a=1$, we get

$$
\begin{aligned}
&P^{(D_n)}_{(r)}(x\,|\,q,\,t)\\
&=\frac{(q;\,q)_r}{(t;\,q)_r}\sum_{0\leqslant 2j\leqslant r}G_{r-2j}(x;\,q,\,t)\frac{(t^{-n}q^{-r};\,q)_j(1/t;\,q)_j}{(t^{-n+1}q^{-r+1};\,q)_j(q;\,q)_j}\frac{1-t^{-n}q^{-r+2j}}{1-t^{-n}q^{-r}}(t^2/q)^j.
\end{aligned}
$$

Hence we have proved Lassalle's conjecture for type D and B [5, p. 10, Conjecture 3 and p. 11, Conjecture 4] (see Section 6.6).

6. Appendix

We recall briefly some facts concerning the Koornwinder polynomials needed for the construction given in Section 5.

6.1. Kernel function $\Pi(x;\, y)$. Let $(a,\, b,\, c,\, d;\, q,\, t)$ be a set of complex parameters with $|q| < 1$. Set $\alpha = (abcdq^{-1})^{1/2}$ for simplicity. Let $x = (x_1,\, \dots,\, x_n)$ be a set of independent indeterminates. Koornwinder's q-difference operator $\mathcal{D}_x = \mathcal{D}_x(a,\, b,\, c,\, d\,|\, q,\, t)$ is defined by [4]

$$\begin{aligned}\mathcal{D}_x = \sum_{i=1}^{n} &\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{\alpha t^{n-1}(1-x_i^2)(1-qx_i^2)} \\ &\qquad\times \prod_{j\neq i} \frac{(1-tx_ix_j)(1-tx_i/x_j)}{(1-x_ix_j)(1-x_i/x_j)}(T_{q,x_i}-1) \\ + \sum_{i=1}^{n} &\frac{(1-a/x_i)(1-b/x_i)(1-c/x_i)(1-d/x_i)}{\alpha t^{n-1}(1-1/x_i^2)(1-q/x_i^2)} \\ &\qquad\times \prod_{j\neq i} \frac{(1-tx_j/x_i)(1-t/x_ix_j)}{(1-x_j/x_i)(1-1/x_ix_j)}(T_{q^{-1},x_i}-1), \end{aligned} \tag{6.1}$$

where $T_{q^{\pm1},x}f(x_1,\, \dots,\, x_i,\, \dots,\, x_n) = f(x_1,\, \dots,\, q^{\pm1}x_i,\, \dots,\, x_n)$. Koornwinder polynomial $P_\lambda(x) = P_\lambda(x\,|\, a,\, b,\, c,\, d\,|\, q,\, t)$ with partition $\lambda = (\lambda_1,\, \dots,\, \lambda_n)$ (i.e., $\lambda_i \in \mathbb{Z}_{\geqslant 0}$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n$) is uniquely characterized by the two conditions (a) $P_\lambda(x)$ is a $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ invariant Laurent polynomial having the triangular expansion in terms of the monomial basis (m_λ) as $P_\lambda(x) = m_\lambda(x) + \text{lower terms}$, (b) $P_\lambda(x)$ satisfies $\mathcal{D}_x P_\lambda = d_\lambda P_\lambda$. The eigenvalue is given by

$$d_\lambda = \sum_{j=1}^{n} \langle abcdq^{-1}t^{2n-2j}q^{\lambda_j}\rangle\langle q^{\lambda_j}\rangle = \sum_{j=1}^{n} \langle \alpha t^{n-j}q^{\lambda_j};\, \alpha t^{n-j}\rangle, \tag{6.2}$$

where we used the notation $\langle x\rangle = x^{1/2} - x^{-1/2}$ and $\langle x;\, y\rangle = \langle xy\rangle\langle x/y\rangle = x + x^{-1} - y - y^{-1}$ for simplicity of display.

Definition 6.1. Define the involution $\widetilde{*}$ of the parameters by

$$\widetilde{a} = \frac{\sqrt{qt}}{a}, \quad \widetilde{b} = \frac{\sqrt{qt}}{b}, \quad \widetilde{c} = \frac{\sqrt{qt}}{c}, \quad \widetilde{d} = \frac{\sqrt{qt}}{d}, \quad \widetilde{q} = q, \quad \widetilde{t} = t. \tag{6.3}$$

We write $\widetilde{\mathcal{D}}_x = \mathcal{D}_x(\widetilde{a},\, \widetilde{b},\, \widetilde{c},\, \widetilde{d}\,|\,\widetilde{q},\, \widetilde{t})$ and $\widetilde{P}_\lambda(x) = P_\lambda(x\,|\,\widetilde{a},\, \widetilde{b},\, \widetilde{c},\, \widetilde{d}\,|\,\widetilde{q},\, \widetilde{t})$ for simplicity of display.

Proposition 6.1. *Let n and m be positive integers, and let $x = (x_1,\, \dots,\, x_n)$, $y = (y_1,\, \dots,\, y_m)$ be two sets of independent indeterminates. Let $\beta \in \mathbb{C}$ be satisfying $t = q^\beta$. Set*

$$\Pi(x;\, y) = \prod_{k=1}^{m} y_k^{\beta n} \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{(q^{1/2}t^{1/2}y_jx_i;\, q)_\infty}{(q^{1/2}t^{-1/2}y_jx_i;\, q)_\infty} \frac{(q^{1/2}t^{1/2}y_j/x_i;\, q)_\infty}{(q^{1/2}t^{-1/2}y_j/x_i;\, q)_\infty}. \tag{6.4}$$

Then we have the kernel function identity (see [3])

$$\mathcal{D}_x\Pi(x;\, y) - \widetilde{\mathcal{D}}_y\Pi(x;\, y) = -\frac{1}{\langle t\rangle}\langle t^n\rangle\langle t^m\rangle\langle abcdq^{-1}t^{n-m-1}\rangle\Pi(x;\, y). \tag{6.5}$$

6.2. Series $f(x; s)$ for BC_n. Let $s = (s_1, \ldots, s_n)$ be a set of complex parameters. Corresponding to s, we introduce $\lambda = (\lambda_1, \ldots, \lambda_n)$ by the conditions $s_i = t^{-n+i}q^{-\lambda_i}$ $(i = 1, \ldots, n)$. We use the notation for the multiple index as $x^{-\lambda} = \prod_i x_i^{-\lambda_i}$. Let $f(x; s) \in x^{-\lambda}\mathbb{C}[[x_1/x_2, \ldots, x_{n-1}/x_n, x_n]]$ be the infinite series satisfying the conditions

$$f(x; s) = x^{-\lambda} \sum_{\alpha \in Q^+} c_\alpha(s) x^\alpha, \quad c_0(s) = 1, \tag{6.6}$$

$$\mathcal{D}_x f(x; s) = \sum_{i=1}^{n} \langle \alpha s_i^{-1}; \alpha t^{n-1} \rangle f(x; s). \tag{6.7}$$

Here Q^+ denotes the positive octant of the root lattice of type BC_n, where the simple roots correspond to the monomials $x_1/x_2, \ldots, x_{n-1}/x_n$ and x_n.

6.3. Reproduction formula. Let $\mathcal{D}_x^*$ be the adjoint of $\mathcal{D}_x$ given by

$$\begin{aligned}\mathcal{D}_x^* = &\sum_{i=1}^{n} (T_{q,x_i}^{-1} - 1) \frac{(1 - ax_i)(1 - bx_i)(1 - cx_i)(1 - dx_i)}{\alpha t^{n-1}(1 - x_i^2)(1 - qx_i^2)} \\ &\qquad \times \prod_{j \neq i} \frac{(1 - tx_ix_j)(1 - tx_i/x_j)}{(1 - x_ix_j)(1 - x_i/x_j)} \\ &+ \sum_{i=1}^{n} (T_{q,x_i} - 1) \frac{(1 - a/x_i)(1 - b/x_i)(1 - c/x_i)(1 - d/x_i)}{\alpha t^{n-1}(1 - 1/x_i^2)(1 - q/x_i^2)} \\ &\qquad \times \prod_{j \neq i} \frac{(1 - tx_j/x_i)(1 - t/x_ix_j)}{(1 - x_j/x_i)(1 - 1/x_ix_j)}.\end{aligned} \tag{6.8}$$

Denote by $V(x)$ the Weyl denominator of type BC_n

$$V(x) = \prod_{k=1}^{n} x_k^{-n+k-1} \prod_{i=1}^{n} (1 - x_i^2) \prod_{1 \leqslant i < j \leqslant n} (1 - x_ix_j)(1 - x_i/x_j). \tag{6.9}$$

Definition 6.2. Define the involution $\overline{*}$ of the parameters by

$$\overline{a} = q/a, \ \overline{b} = q/b, \ \overline{c} = q/c, \ \overline{d} = q/d, \ \overline{q} = q, \ \overline{t} = q/t. \tag{6.10}$$

Write for simplicity the composition of the involutions as $\widehat{*} = \overline{\overline{*}}$, namely we have

$$\widehat{a} = \sqrt{q/t}\, a, \ \widehat{b} = \sqrt{q/t}\, b, \ \widehat{c} = \sqrt{q/t}\, c, \ \widehat{d} = \sqrt{q/t}\, d, \ \widehat{q} = q, \ \widehat{t} = q/t. \tag{6.11}$$

Proposition 6.2. *We have*

$$V(x)^{-1} \mathcal{D}_x^* V(x) - \overline{\mathcal{D}}_x = \sum_{j=1}^{n} \langle \overline{\alpha}\, \overline{t}^{n-j}; \alpha t^{n-j} \rangle. \tag{6.12}$$

Theorem 6.1. *Let $n \geqslant m$ be positive integers, and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ be sets of variables. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of length $\leqslant m$. Set*

$$s_i = \widehat{t}^{-m+i} q^{-\lambda_{m+1-i}+m+1-i-n\beta} \quad (1 \leqslant i \leqslant m). \tag{6.13}$$

Let $\widehat{f}(y;\, s)$ denote the formal series in y characterized by

$$\widehat{f}(y;\, s) = \prod_{i=1}^{m} y_i^{-\lambda_{m+1-i}+m+1-i-n\beta} \sum_{\alpha\in Q^+} \widehat{c}_\alpha(s) y^\alpha, \tag{6.14}$$

$$\widehat{\mathcal{D}}_y \widehat{f}(y;\, s) = \sum_{i=1}^{m} \langle \widehat{\alpha} s_i^{-1};\, \widehat{\alpha}\widehat{t}^{m-i}\rangle \widehat{f}(y;\, s). \tag{6.15}$$

Then $\Pi(x;\, y)V(y)\widehat{f}(y;\, s)$ has no fractional powers in y, allowing us to consider the constant term in y. We have

$$\mathcal{D}_x \left[\Pi(x;\, y)V(y)\widehat{f}(y;\, s)\right]_{1,y} = \sum_{i=1}^{m} \langle \alpha t^{n-1} q^{\lambda_i};\, \alpha t^{n-i}\rangle \left[\Pi(x;\, y)V(y)\widehat{f}(y;\, s)\right]_{1,y} \tag{6.16}$$

(the symbol $[\cdots]_{1,y}$ denotes the constant term in y). Hence $[\Pi(x;\, y)V(y)\widehat{f}(y;\, s)]_{1,y}$ gives us the Koormwinder polynomila $P_\lambda(x)$ up to a multiplication constant.

Proof. Note that from the choice of s (6.13) and the definition of $\Pi(x;\, y)$ in (6.4), fractional powers in y cancel in the combination $\Pi(x;\, y)V(y)\widehat{f}(y;\, s)$. We have

$$\begin{aligned}
&\left(\mathcal{D}_x + \frac{\langle t^n\rangle\langle t^m\rangle\langle abcdq^{-1}t^{n-m-1}\rangle}{\langle t\rangle}\right)\left[\Pi(x;\, y)V(y)\widehat{f}(y;\, s)\right]_{1,y} \\
&\qquad = \left[\big(\widetilde{\mathcal{D}}_y \Pi(x;\, y)\big)V(y)\widehat{f}(y;\, s)\right]_{1,y} \\
&\qquad = \left[\Pi(x;\, y)\big(\widetilde{\mathcal{D}}_x^* V(y)\widehat{f}(y;\, s)\big)\right]_{1,y} \\
&\qquad = \left[\Pi(x;\, y)V(y)\left(\left(\widehat{\mathcal{D}}_x + \sum_{i=1}^{m} \langle \widehat{\alpha}\widehat{t}^{m-i};\, \widetilde{\alpha} t^{m-i}\rangle\right)\widehat{f}(y;\, s)\right)\right]_{1,y}.
\end{aligned} \tag{6.17}$$

To calculate the eigenvalue, we need the following result.

Lemma 6.1. *Write $\alpha = (abcdq^{-1})^{1/2}$, $\widetilde{\alpha} = t/\alpha$. We have*

$$\frac{1}{\langle t\rangle}\langle t^n\rangle\langle t^m\rangle\langle abcdq^{-1}t^{n-m-1}\rangle = \sum_{i=1}^{m\wedge n} \langle \alpha t^{n-i};\, \widetilde{\alpha} t^{m-i}\rangle. \tag{6.18}$$

Hence by denoting $\widehat{\alpha} = q\alpha/t$ and (6.13) we have

$$\begin{aligned}
&\sum_{i=1}^{m} \langle \widehat{\alpha} s_i^{-1};\, \widehat{\alpha}\widehat{t}^{m-i}\rangle + \sum_{i=1}^{m} \langle \widehat{\alpha}\widehat{t}^{m-i};\, \widetilde{\alpha} t^{m-i}\rangle - \frac{1}{\langle t\rangle}\langle t^n\rangle\langle t^m\rangle\langle abcdq^{-1}t^{n-m-1}\rangle \\
&= \sum_{i=1}^{m} \langle \widehat{\alpha} s_i^{-1};\, \alpha t^{n-i}\rangle = \sum_{i=1}^{m} \langle \alpha t^{n-m-1+i} q^{\lambda_{m+1-i}};\, \alpha t^{n-i}\rangle = \sum_{i=1}^{m} \langle \alpha t^{n-i} q^{\lambda_i};\, \alpha t^{n-i}\rangle.
\end{aligned} \tag{6.19}$$

□

6.4. Macdonald polynomials of type C. We consider some degenerations of the Koornwinder polynomials to Macdonald polynomials. As for the details, we refer the reader to [4] and [6]. Setting the parameters as

$$(a,\, b,\, c,\, d,\, q,\, t) \to (-b^{1/2},\, ab^{1/2},\, -q^{1/2}b^{1/2},\, q^{1/2}ab^{1/2},\, q,\, t)$$

in the Koornwinder polynomial $P_\lambda(x)$, we obtain the Macdonald polynomials of type $(BC_n,\, C_n)$ (see [4])

$$P_\lambda^{(BC_n,C_n)}(x\,|\,a,\, b;\, q,\, t) = P_\lambda(x\,|-b^{1/2},\, ab^{1/2},\, -q^{1/2}b^{1/2},\, q^{1/2}ab^{1/2}\,|\,q,\, t) \tag{6.20}$$

Namely, setting

$$D_x^{(BC_n,C_n)} = \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \prod_{i=1}^{n} \frac{(1-ab^{1/2}x_i^{\sigma_i})(1+b^{1/2}x_i^{\sigma_i})}{1-x_i^{2\sigma_i}} \times \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_i^{\sigma_i}x_i^{\sigma_j}}{1-x_i^{\sigma_i}x_i^{\sigma_j}} T_{q^{\sigma_1/2},x_1}\cdots T_{q^{\sigma_n/2},x_n}, \tag{6.21}$$

we have

$$P_\lambda^{(BC_n,C_n)}(x) = m_\lambda + \text{lower terms}, \tag{6.22}$$

$$D_x^{(BC_n,C_n)} P_\lambda^{(BC_n,C_n)}(x) = (ab)^{n/2} t^{n(n-1)/4} \sum_{\sigma_1,\cdots,\sigma_n=\pm 1} s_1^{\sigma_1/2}\cdots s_n^{\sigma_n/2}\cdot P_\lambda^{(BC_n,C_n)}(x), \tag{6.23}$$

where $s_i = abt^{n-i}q^{\lambda_i}$.

The special case $a = 1$ is called the Macdonald polynomials of type C_n

$$P_\lambda^{(C_n)}(x\,|\,b;\, q,\, t) = P_\lambda^{(BC_n,C_n)}(x\,|\,1,\, b;\, q,\, t). \tag{6.24}$$

Note that the twist $\widehat{*}$ with the parameters $(-b^{1/2}, ab^{1/2}, -q^{1/2}b^{1/2}, q^{1/2}ab^{1/2}, q, t)$ gives

$$\big(-(qb/t)^{1/2},\, a(qb/t)^{1/2},\, -q^{1/2}(qb/t)^{1/2},\, q^{1/2}a(qb/t)^{1/2};\, q,\, q/t\big). \tag{6.25}$$

6.5. Macdonald polynomials of type B and D. Setting the parameters as $(a,\, b,\, c,\, d,\, q,\, t) \to (-b^{1/2},\, ab^{1/2},\, -q^{1/2},\, q^{1/2};\, q,\, t)$ in the Koornwinder polynomial $P_\lambda(x)$, we obtain the Macdonald polynomials of type $(BC_n,\, B_n)$ (see [4])

$$P_\lambda^{(BC_n,B_n)}(x\,|\,a,\, b;\, q,\, t) = P_\lambda(x\,|-b^{1/2},\, ab^{1/2},\, -q^{1/2},\, q^{1/2}\,|\,q,\, t). \tag{6.26}$$

Setting $b = 1$, we get the Macdonald polynomial of type B_n

$$P_\lambda^{(B_n)}(x\,|\,a;\, q,\, t) = P_\lambda^{(BC_n,B_n)}(x\,|\,a,\, 1;\, q,\, t). \tag{6.27}$$

Setting further $a = 1$, we get the Macdonald polynomial of type D_n

$$P_\lambda^{(D_n)}(x\,|\,q,\, t) = P_\lambda^{(BC_n,B_n)}(x\,|\,1,\, 1;\, q,\, t). \tag{6.28}$$

The application of the twist $\widehat{*}$ on the parameters $(-b^{1/2},\, ab^{1/2},\, -q^{1/2},\, q^{1/2};\, q,\, t)$ gives

$$(-(qb/t)^{1/2},\, a(qb/t)^{1/2},\, -qt^{-1/2},\, qt^{-1/2};\, q,\, q/t). \tag{6.29}$$

6.6. Lassalle's conjectures. For the readers' convenience, we recall Lassalle's conjectures for Macdonald polynomials of type B, C and D with one row diagram.

Set

$$\begin{aligned} g_r^{(C_n)}(x) &= \frac{(t;\,q)_r}{(q;\,q)_r} P_{(r)}^{(C_n)}(x\,|\,b;\,q,\,t) \\ &= \frac{(t;\,q)_r}{(q;\,q)_r} P_{(r)}(x\,|-b^{1/2},\,b^{1/2},\,-q^{1/2}b^{1/2},\,q^{1/2}b^{1/2}\,|\,q,\,t), \end{aligned} \tag{6.30}$$

$$\begin{aligned} g_r^{(B_n)}(x) &= \frac{(t;\,q)_r}{(q;\,q)_r} P_{(r)}^{(B_n)}(x\,|\,a;\,q,\,t) \\ &= \frac{(t;\,q)_r}{(q;\,q)_r} P_{(r)}(x\,|-1,\,a,\,-q^{1/2},\,q^{1/2}\,|\,q,\,t), \end{aligned} \tag{6.31}$$

$$\begin{aligned} g_r^{(D_n)}(x) &= \frac{(t;\,q)_r}{(q;\,q)_r} P_{(r)}^{(D_n)}(x\,|\,q,\,t) \\ &= \frac{(t;\,q)_r}{(q;\,q)_r} P_{(r)}(x\,|-1,\,1,\,-q^{1/2},\,q^{1/2}\,|\,q,\,t). \end{aligned} \tag{6.32}$$

Lassalle's conjectures [5, p.8, Conjecture 1, p.10, Conjecture 3, p.11, Conjecture 4] read

$$g_r^{(C_n)} = \sum_{i=0}^{\lfloor r/2\rfloor} G_{r-2i} t^i \frac{(b/t;\,q)_i}{(q;\,q)_i} \frac{(t^n q^{r-i};\,q)_i}{(bt^{n-1}q^{r-i};\,q)_i} \frac{1-t^n q^{r-2i}}{1-t^n q^{r-i}}, \tag{6.33}$$

$$g_r^{(D_n)} = \sum_{i=0}^{\lfloor r/2\rfloor} G_{r-2i} t^i \frac{(1/t;\,q)_i}{(q;\,q)_i} \frac{(t^n q^{r-i};\,q)_i}{(t^{n-1}q^{r-i};\,q)_i} \frac{1-t^n q^{r-2i}}{1-t^n q^{r-i}}, \tag{6.34}$$

$$g_r^{(B_n)} = \sum_{i=0}^{r} g_{r-i}^{(D_n)} \frac{(a;\,q)_i}{(q;\,q)_i} \frac{(t^n q^{r-i};\,q)_i}{(t^{n-1}q^{r-i};\,q)_i} \frac{(t^{2n-2}q^{2r-i+1};\,q)_i}{(at^{2n-2}q^{2r-i};\,q)_i}. \tag{6.35}$$

Namely, for type C and D we have

$$g_r^{(C_n)} = \sum_{i=0}^{\lfloor r/2\rfloor} G_{r-2i} \frac{(b/t;\,q)_i}{(q;\,q)_i} \frac{(t^{-n}q^{-r};\,q)_i}{(t^{-n+1}q^{-r+1}/b;\,q)_i} \frac{1-t^{-n}q^{-r+2i}}{1-t^{-n}q^{-r}} (t^2/qb)^i, \tag{6.36}$$

$$g_r^{(D_n)} = \sum_{i=0}^{\lfloor r/2\rfloor} G_{r-2i} \frac{(1/t;\,q)_i}{(q;\,q)_i} \frac{(t^{-n}q^{-r};\,q)_i}{(t^{-n+1}q^{-r+1};\,q)_i} \frac{1-t^{-n}q^{-r+2i}}{1-t^{-n}q^{-r}} (t^2/q)^i. \tag{6.37}$$

For type B we have

$$\begin{aligned} g_r^{(B_n)} &= \sum_{i=0}^{r} g_{r-i}^{(D_n)} \frac{(a;\,q)_i}{(q;\,q)_i} \frac{(t^{-n}q^{-r+1};\,q)_i}{(t^{-n+1}q^{-r};\,q)_i} \frac{(t^{-2n+2}q^{-2r};\,q)_i}{(t^{-2n+2}q^{-2r+1}/a;\,q)_i} (t/a)^i \\ &= \sum_{i=0}^{r} \sum_{j=0}^{\lfloor (r-i)/2\rfloor} G_{r-i-2j} \frac{(a;\,q)_i}{(q;\,q)_i} \frac{(t^{-n}q^{-r+1};\,q)_i}{(t^{-n+1}q^{-r};\,q)_i} \frac{(t^{-2n+2}q^{-2r};\,q)_i}{(t^{-2n+2}q^{-2r+1}/a;\,q)_i} (t/a)^i \\ &\quad \times \frac{(1/t;\,q)_j}{(q;\,q)_j} \frac{(t^{-n}q^{-r+i};\,q)_j}{(t^{-n+1}q^{-r+i+1};\,q)_j} \frac{1-t^{-n}q^{-r+i+2j}}{1-t^{-n}q^{-r+i}} (t^2/q)^j \end{aligned}$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{\lfloor (r-i)/2 \rfloor} G_{r-i-2j} \frac{(a;\,q)_i}{(q;\,q)_i} \frac{(t^{-n+1}q^{-r+1};\,q)_i}{(t^{-n+1}q^{-r};\,q)_i} \frac{(t^{-2n+2}q^{-2r};\,q)_i}{(t^{-2n+2}q^{-2r+1}/a;\,q)_i} (t/a)^i \\ \times \frac{(1/t;\,q)_j}{(q;\,q)_j} \frac{(t^{-n}q^{-r};\,q)_{i+j}}{(t^{-n+1}q^{-r+1};\,q)_{i+j}} \frac{1-t^{-n}q^{-r+i+2j}}{1-t^{-n}q^{-r}} (t^2/q)^j. \quad (6.38)$$

6.7. Conjecture about the Macdonald polynomial of type B_2. We present a conjecture about the formal series $f(x;\,s)$ for type B_2. Let ε_1, ε_2 be the standard basis for $\mathbb{R}^2$. The simple roots (for B_2) are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2$, fundamental weights are $\omega_1 = \varepsilon_1$, $\omega_2 = (\varepsilon_1 + \varepsilon_2)/2$. Let P and P^+ be the weight lattice and the cone of dominant weights. Let W be the Weyl group of type B_2. Denote the monomial symmetric polynomials by m_λ ($\lambda \in P^+$): $m_\lambda = \sum_{\mu \in W^\lambda} e^\mu$. We write $x_1 = e^{\varepsilon_1}$, $x_2 = e^{\varepsilon_2}$ for simplicity.

Let q, t and T be indeterminates. The Macdonald difference operator for type B_2 is defined by

$$E_{\omega_1}(q,\,t,\,T) = \frac{1-tx_1/x_2}{1-x_1/x_2} \frac{1-tx_1x_2}{1-x_1x_2} \frac{1-Tx_1}{1-x_1} T_{q,x_1} \\ + \frac{1-tx_2/x_1}{1-x_2/x_1} \frac{1-tx_1x_2}{1-x_1x_2} \frac{1-Tx_2}{1-x_2} T_{q,x_2} + \frac{1-t/x_1x_2}{1-1/x_1x_2} \frac{1-tx_2/x_1}{1-x_2/x_1} \frac{1-T/x_1}{1-1/x_1} T_{q^{-1},x_1} \\ + \frac{1-t/x_1x_2}{1-1/x_1x_2} \frac{1-tx_1/x_2}{1-x_1/x_2} \frac{1-T/x_2}{1-1/x_2} T_{q^{-1},x_2}. \quad (6.39)$$

The Macdonald polynomials $P_\lambda(x_1,\,x_2;\,q,\,t,\,T)$ of type B_2 are uniquely characterized by the following conditions.

$$\text{(i)}\ \ P_\lambda(x_1,\,x_2;\,q,\,t,\,T) = m_\lambda + \sum_{\mu \in P^+, \mu < \lambda} a_{\lambda\mu}(q,\,t,\,T) m_\mu, \quad (6.40)$$

$$\text{(ii)}\ \ E_{\omega_1}(q,\,t,\,T) P_\lambda(x_1,\,x_2;\,q,\,t,\,T) = c_\lambda P_\lambda(x_1,\,x_2;\,q,\,t,\,T), \quad (6.41)$$

where

$$c_{r_1\omega_1 + r_2\omega_2} = t^2Tq^{r_1+r_2/2} + tTq^{r_2/2} + tq^{-r_2/2} + q^{-r_1-r_2/2} \quad (r_1,\,r_2 \in \mathbb{Z}_{\geqslant 0}), \quad (6.42)$$

or equivalently

$$c_{\lambda_1\varepsilon_1+\lambda_2\varepsilon_2} = t^2Tq^{\lambda_1} + tTq^{\lambda_2} + tq^{-\lambda_2} + q^{-\lambda_1} \\ (\lambda_1,\,\lambda_2 \in \mathbb{Z}_{\geqslant 0}/2,\ \lambda_1+\lambda_2 \in \mathbb{Z}_{\geqslant 0},\ \lambda_1 \geqslant \lambda_2). \quad (6.43)$$

Let s_1, s_2 be generic parameters. Introduce variables λ_1, λ_2 satisfying $s_1 = tT^{1/2}q^{\lambda_1}$, $s_2 = T^{1/2}q^{\lambda_2}$. Note that we have $T_{q,x_1}x_1^{\lambda_1} = t^{-1}T^{-1/2}s_1x_1^{\lambda_1}$, $T_{q,x_2}x_2^{\lambda_2} = T^{-1/2}s_2x_2^{\lambda_2}$.

Set

$$f^{B_2}(x_1,\,x_2;\,s_1,\,s_2,\,q,\,t,\,T) = x_1^{\lambda_1}x_2^{\lambda_2} \sum_{n=0}^{\infty} f_n^{B_2}(x_1,\,x_2;\,s_1,\,s_2,\,q,\,t,\,T), \quad (6.44)$$

$$\begin{aligned}
&f_n^{B_2}(x_1, x_2; s_1, s_2, q, t, T)\\
&\quad= (q^2t/T^2)^n \frac{(q/t; q)_n (T/t; q)_n (T; q)_n (T/s_1s_2; q)_n (T/ts_1s_2; q)_n (q/ts_1s_2; q)_n}{(q; q)_n (q/s_1^2; q)_n (q/s_2^2; q)_n (qs_1/s_2; q)_n (qs_2/s_1; q)_n (q/s_1s_2; q)_n}\\
&\qquad\qquad\times \frac{(T/s_1^2; q)_{2n} (T/s_2^2; q)_{2n}}{(T/s_1s_2; q)_{2n} (T/ts_1s_2; q)_{2n}} (1/x_1x_2)^n\\
&\qquad\quad\times \sum_{\theta_1,\theta_2,\theta_3,\theta_4\geqslant 0} c^{B_2}(n, \theta_1, \theta_2, \theta_3, \theta_4; s_1, s_2, q, t, T)\\
&\qquad\qquad\qquad\times (x_2/x_1)^{\theta_1} (1/x_2)^{\theta_2} (1/x_1)^{\theta_3} (1/x_1x_2)^{\theta_4}, \qquad (6.45)
\end{aligned}$$

where

$$\begin{aligned}
&c^{B_2}(n, \theta_1, \theta_2, \theta_3, \theta_4; s_1, s_2, q, t, T) = (q/t)^{\theta_1} \frac{(t; q)_{\theta_1}}{(q; q)_{\theta_1}} \frac{(q^{\theta_3-\theta_2}ts_2/s_1; q)_{\theta_1}}{(q^{\theta_3-\theta_2}qs_2/s_1; q)_{\theta_1}}\\
&\quad\times (q/T)^{\theta_2} \frac{(q^nT; q)_{\theta_2}}{(q; q)_{\theta_2}} \frac{(q^{2n}T/s_2^2; q)_{\theta_2}}{(q^nq/s_2^2; q)_{\theta_2}} \frac{(q^nT/s_1s_2; q)_{\theta_2}}{(q^{2n}T/s_1s_2; q)_{\theta_2}} \frac{(qs_1/s_2; q)_{\theta_2}}{(q^nqs_1/s_2; q)_{\theta_2}}\\
&\quad\times (q/T)^{\theta_3} \frac{(q^nT; q)_{\theta_3}}{(q; q)_{\theta_3}} \frac{(q^{2n}T/s_1^2; q)_{\theta_3}}{(q^nq/s_1^2; q)_{\theta_3}} \frac{(ts_2/s_1; q)_{\theta_3}}{(q^nqs_2/s_1; q)_{\theta_3}} \frac{(q^{-\theta_2}qs_2/ts_1; q)_{\theta_3}}{(q^{-\theta_2}s_2/s_1; q)_{\theta_3}}\\
&\quad\times \frac{(q^nT/s_1s_2; q)_{\theta_3}}{(q^{2n}qT/ts_1s_2; q)_{\theta_3}} \frac{(q^{2n}q^{\theta_2}qT/ts_1s_2; q)_{\theta_3}}{(q^{2n}q^{\theta_2}T/s_1s_2; q)_{\theta_3}}\\
&\quad\times (q/t)^{\theta_4} \frac{(t; q)_{\theta_4}}{(q; q)_{\theta_4}} \frac{(q^{2n}q^{\theta_2+\theta_3}tT/s_1s_2; q)_{\theta_4}}{(q^{2n}q^{\theta_2+\theta_3}qT/s_1s_2; q)_{\theta_4}}. \qquad (6.46)
\end{aligned}$$

Conjecture 6.1. *The series $f^{B_2}(x_1, x_2; s_1, s_2, q, t, T)$ in (6.44) satisfies*

$$\begin{aligned}
&E_{\omega_1}(q, t, T) f^{B_2}(x_1, x_2; s_1, s_2, q, t, T)\\
&\qquad= tT^{1/2}(s_1 + s_2 + s_1^{-1} + s_2^{-1}) f^{B_2}(x_1, x_2; s_1, s_2, q, t, T). \qquad (6.47)
\end{aligned}$$

Conjecture 6.2. *Let $r_1, r_2 \in \mathbb{Z}_{\geqslant 0}$. We have*

$$\begin{aligned}
&P_{r_1\omega_1+r_2\omega_2}(x_1, x_2; q, t, T)\\
&\qquad= x_1^{r_1+r_2/2} x_2^{r_2/2} f^{B_2}(x_1, x_2; tT^{1/2}q^{r_1+r_2/2}, T^{1/2}q^{r_2/2}, q, t, T). \qquad (6.48)
\end{aligned}$$

Or equivalently, for any half-partition (λ_1, λ_2) (i.e., $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geqslant 0}$ or $1/2 + \mathbb{Z}_{\geqslant 0}$ simultaneously, and $\lambda_1 \geqslant \lambda_2$), we have

$$P_{\lambda_1\varepsilon_1+\lambda_2\varepsilon_2}(x_1, x_2; q, t, T) = x_1^{\lambda_1} x_2^{\lambda_2} f^{B_2}(x_1, x_2; tT^{1/2}q^{\lambda_1}, T^{1/2}q^{\lambda_2}, q, t, T). \qquad (6.49)$$

Remark 6.1. 1) When the parameters are specialized as

$$s_1 = tT^{1/2}q^{r_1+r_2/2}, \quad s_2 = T^{1/2}q^{r_2/2} \quad (r_1, r_2 \in \mathbb{Z}_{\geqslant 0}). \qquad (6.50)$$

the series $f^{B_2}(x_1, x_2; s_1, s_2, q, t, T)$ becomes truncated. It also has to be symmetric with respect to the action of the Weyl group of type B_2. The truncation can be checked explicitly from (6.46). We have not checked the symmetry yet.

2) If $r = r\omega_1$ ($s_1 = tT^{1/2}q^r$, $s_2 = T^{1/2}$, $r \in \mathbb{Z}_{\geqslant 0}$, $r_2 = 0$), in view of (6.45), we have $f_n^{B_2}(x_1, x_2; s_1, s_2, q, t, T) = 0$ when $n > 0$. Hence Conjecture 6.2 implies the threefold summation formula

$$\begin{aligned} P_{r\omega_1}(x_1, x_2; q, t, T) &= x_1^r f_0^{B_2}(x_1, x_2; tT^{1/2}q^r, T^{1/2}, q, t, T) \\ &= x_1^r \sum_{\theta_1,\theta_3,\theta_4 \geqslant 0} (x_2/x_1)^{\theta_1}(1/x_1)^{\theta_3}(1/x_1x_2)^{\theta_4} \\ \times (q/t)^{\theta_1} &\frac{(t;q)_{\theta_1}}{(q;q)_{\theta_1}} \frac{(q^{\theta_3-r};q)_{\theta_1}}{(q^{\theta_3-r+1}/t;q)_{\theta_1}} (q/T)^{\theta_3} \frac{(T;q)_{\theta_3}}{(q;q)_{\theta_3}} \frac{(q^{-2r}/t^2;q)_{\theta_3}}{(q^{-2r+1}/t^2T;q)_{\theta_3}} \\ \times &\frac{(q^{-r};q)_{\theta_3}}{(q^{-r+1}/t;q)_{\theta_3}} \frac{(q^{-r+1}/t^2;q)_{\theta_3}}{(q^{-r}/t;q)_{\theta_3}} (q/t)^{\theta_4} \frac{(t;q)_{\theta_4}}{(q;q)_{\theta_4}} \frac{(q^{\theta_3-r};q)_{\theta_4}}{(q^{\theta_3-r+1}/t;q)_{\theta_4}}. \end{aligned} \tag{6.51}$$

3) Let $\lambda = r_1\omega_1 + r_2\omega_2$ ($r_1, r_2 \in \mathbb{Z}_{\geqslant 0}$), and s_1, s_2 are specialized as in (6.50). If we further specialize the parameters as $q = t = T$, the irreducible character $\mathrm{ch}_{r_1\omega_1+r_2\omega_2}$ for the Lie algebra of type B_2 must be recovered from Conjecture 6.2. By checking termination of the series, one finds that Conjecture 6.2 implies

$$\mathrm{ch}_{r_1\omega_1+r_2\omega_2} = x_1^{r_1+r_2/2}x_2^{r_2/2} \sum_{\substack{\theta_1,\theta_2,\theta_3,\theta_4\geqslant 0 \\ \theta_1-\theta_2+\theta_3\leqslant r_1 \\ \theta_2\leqslant r_2,\, \theta_3\leqslant r_1 \\ \theta_2+\theta_3+\theta_4\leqslant r_1+r_2}} (x_2/x_1)^{\theta_1}(1/x_2)^{\theta_2}(1/x_1)^{\theta_3}(1/x_1x_2)^{\theta_4}. \tag{6.52}$$

4) Conjecture 6.2 has been checked up to $r_1 + r_2 \leqslant 6$.

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