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PIVOTAL FUSION CATEGORIES OF RANK 3

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Dedicated to Boris Feigin with admiration

Abstract. We classify all fusion categories of rank 3 that admit a pivotal structure over an algebraically closed field of characteristic zero.

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1. Introduction

Fusion categories introduced in [ENO1] are in a sense simplest tensor categories. The problem of classification of fusion categories while interesting seems to be well out of reach at the moment. Thus it is natural to attempt a classification of fusion categories which are "small" in certain sense. One classical interpretation of "small" coming from the theory of subfactors is that the category in question contains an algebra object of small dimension; we refer the reader to [JMS] for a nice survey of achievements in this direction. Another possibility is that "small" categories have integral Frobenius–Perron dimension with small number of prime factors; some results in this direction were obtained in [ENO2].

In this paper we follow the approach initiated in [O2], in which "small" categories have small *rank*. We recall that rank of a fusion category is just a number of isomorphism classes of its simple objects. The fusion categories of rank 1 are trivial and the fusion categories of rank 2 were classified in [O2]. Under an additional assumption that the category in question is ribbon some further classification results were obtained in [O3], [RSW], [HR]. Also recently a general finiteness conjecture on the number of modular tensor categories of fixed rank was proved, see [BNRW].

In this paper we give a classification of fusion categories of rank 3 that admit a pivotal structure (this is relatively mild assumption: it is conjectured in [ENO1, Conjecture 2.8] that any fusion category admits a pivotal structure). Thus we prove the following result conjectured in [O3]:

Theorem 1.1. *Let $\mathcal{C}$ be a fusion category of rank 3 admitting a pivotal structure over an algebraically closed field of characteristic zero. Then $\mathcal{C}$ is equivalent to one of the following*:

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(i) *one of three pointed category with underlying group* $\mathbb{Z}/3\mathbb{Z}$ (*corresponding to elements of* $H^3(\mathbb{Z}/3\mathbb{Z}, \mathbb{C}^\times)$), *see for example* [ENO1, p. 584];

(ii) *one of three categories associated with quantum* $\mathfrak{so}_3$ *at* 7*th root of unity, see for example* [O3, Section 4.3]*;*

(iii) *one of two Ising categories, see for example* [DGNO, Appendix B]*;*

(iv) *category of representations of the group* S_3 *or one of two twisted versions of it, see for example* [O3, Section 4.4]*;*

(v) *category associated with subfactor of type* E_6 *or its Galois conjugate, see for example* [HH].

We note that in all cases it is easy to classify all possible pivotal structures using [DGNO, Section 2.4.3].

The proof of Theorem 1.1 is significantly more difficult than the previous classification in [O2] of fusion categories of rank 2. Thus we had to develop some new theoretical tools. Most important are Theorem 2.13, which gives some information on the decomposition of the induction of the unit object to the Drinfeld center of a fusion category, and the pseudo-unitary inequality (5), which gives some non-trivial restrictions on the Grothendieck rings of fusion categories admitting a spherical structure with positive values of dimensions. This inequality is strong enough to show that vast majority of based rings of rank 3 are not categorifiable. It is also sometimes useful in some other cases, see for example [La]. However the computer experiments performed by Nicolle Sandoval Gonzalez seem to indicate that this inequality is less efficient in higher ranks.

It is highly unsatisfactory that we had to assume the existence of pivotal structure for our results. Despite a significant effort we were not able to eliminate this assumption. However the existence of fusion categories of rank 3 which do not admit such a structure seems to be very unlikely: we prove that such categories must have very big fusion coefficients, see Theorem 5.8.

It is my great pleasure to express my deep gratitude to Dmitri Nikshych who made many useful suggestions for this paper. In particular he suggested to use results of Ng and Schauenburg in the proof of Theorem 4.1, which allowed me to eliminate one particularly difficult case. Also Example 2.18 is a result of our joint work. I am happy to thank Nicolle Sandoval Gonzalez and Hannah Larson for many useful discussions. I am also grateful to the anonymous referee for useful comments and showing me reference [GHPS].

2. Preliminaries

In this paper the base field k is assumed to be algebraically closed of characteristic zero.

2.1. Fusion categories and based rings. We recall from [ENO1] that a fusion category is a rigid tensor semisimple category over k such that all Hom spaces are finite dimensional, the number of isomorphism classes of simple objects is finite, and the unit object is simple. For a fusion category $\mathcal{C}$ we will denote by $\mathcal{O}(\mathcal{C})$ the set of isomorphism classes of simple objects in $\mathcal{C}$.

A *pivotal structure* on a fusion category $\mathcal{C}$ is an isomorphism of tensor functors $\mathrm{id} \to {}^{**}$, see for example [ENO1, Definition 2.7]. In the presence of pivotal structure one defines traces of morphisms and dimensions $\dim(X) \in k$ of objects, see for example [ENO1, Section 2.2]. A pivotal structure (or the underlying fusion category) is *spherical* if $\dim(X) = \dim(X^*)$ for all objects $X \in \mathcal{C}$ (equivalently, all dimensions are totally real), see [ENO1, Section 2.2]. For a fusion category $\mathcal{C}$ one defines its *global dimension* $\dim(\mathcal{C})$, see for example [ENO1, Definition 2.2]. For a spherical fusion category $\mathcal{C}$ the global dimension is given by

$$\dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \dim(X)^2.$$

Assume that we fixed an embedding of the subfield of algebraic numbers in k to $\mathbb{C}$. A *pseudo-unitary spherical structure* on a fusion category $\mathcal{C}$ is a spherical structure such that the images under this embedding of dimensions of all objects are positive real numbers, see [ENO1, Section 8.4]. A fusion category is *pseudo-unitary* if it has a pseudo-unitary spherical structure.

The Grothendieck ring $K(\mathcal{C})$ of a fusion category $\mathcal{C}$ is an example of *unital based ring* of finite rank (see for example [O4, Definition 2.1]). This means that $K(\mathcal{C})$ is endowed with a basis $\{b_i\}_{i \in I}$ over $\mathbb{Z}$ (given by the classes of simple objects) such that the structure constants are nonnegative; moreover there exists an involution $i \mapsto i^*$ such that the expansion of $b_i b_j$ contains 1 (which is one of the basis elements) with coefficient δ_{ij^*}.

For a based ring K of finite rank its *categorification* is a fusion category $\mathcal{C}$ and an isomorphism of based rings $K \simeq K(\mathcal{C})$; two categorifications are *equivalent* if there is a tensor equivalence $\mathcal{C}_1 \to \mathcal{C}_2$ inducing the identity map on K. It is known that a given based ring of finite rank admits only finitely many categorifications up to equivalence (this statement is known as "Ocneanu rigidity", see [ENO1, Theorem 2.28]). We say that a based ring is *categorifiable* if it admits at least one categorification. The following statements are useful necessary conditions for a based ring K to be categorifiable:

Proposition 2.1 (Cyclotomic test, see [ENO1, Corollary 8.53])**.** *Let $\mathcal{C}$ be a fusion category. Any irreducible representation of the ring $K(\mathcal{C})$ is defined over some cyclotomic field. In particular for any homomorphism $\phi\colon K(\mathcal{C}) \to \mathbb{C}$ we have $\phi([X]) \in \mathbb{Z}[\zeta]$ for some root of unity ζ.* □

We recall (see [O4]) that an algebraic integer α is a *d-number* if $\frac{\sigma(\alpha)}{\alpha}$ is algebraic integer for any automorphism σ of $\bar{\mathbb{Q}}$. Equivalently, if $x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{Z}[x]$ is the minimal polynomial of α then $\frac{a_i^n}{a_n^i} \in \mathbb{Z}$ for $i = 1, \ldots, n-1$.

Proposition 2.2 (d-number test, see [O4, Theorem 1.2])**.** *Let $\mathcal{C}$ be a fusion category. Let $R\colon K(\mathcal{C}) \to K(\mathcal{C})$ be the operator of left multiplication by the element $\sum_i b_i b_{i^*}$. Then the eigenvalues of R are d-numbers.* □

Remark 2.3. Proposition 2.2 is equivalent to [O4, Theorem 1.2] in view of Remark 2.11 and [O4, Corollary 2.8].

2.2. The induction functor $I\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. Let $\mathcal{C}$ be a spherical fusion category. It is known that its *Drinfeld center* $\mathcal{Z}(\mathcal{C})$ is a *modular tensor category*, see [M2, Theorem 1.2], [ENO1, Theorem 2.15]. Since $\mathcal{Z}(\mathcal{C})$ is semisimple the forgetful functor $F\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint $I\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. We will use the following properties of this functor:

Proposition 2.4 [ENO1, Proposition 5.4]**.** $F(I(Y)) = \bigoplus_{X\in\mathcal{O}(\mathcal{C})} X \otimes Y \otimes X^*$.

In particular we see that for any $Y \in \mathcal{C}$

$$\dim(I(Y)) = \dim(Y)\dim(\mathcal{C}). \tag{1}$$

Recall that any modular tensor category is equipped with balance isomorphisms $\theta_X\colon X \to X$, see [BK, Section 2.2].

Theorem 2.5 (cf. [NS, Remark 4.6])**.** *We have* $\theta_{I(\mathbf{1})} = \mathrm{id}_{I(\mathbf{1})}$ *and* $\mathrm{Tr}(\theta_{I(\mathbf{1})}) = \dim(\mathcal{C})$. *For any* $\mathbf{1} \neq X \in \mathcal{O}(\mathcal{C})$ *we have* $\mathrm{Tr}(\theta_{I(X)}) = 0$.

Proof. The equality $\theta_{I(\mathbf{1})} = \mathrm{id}_{I(\mathbf{1})}$ follows from [O5, Corollary 2.7]; then $\mathrm{Tr}(\theta_{I(\mathbf{1})}) = \dim(I(\mathbf{1})) = \dim(\mathcal{C})$ follows from (1).

Recall (see [DMNO, Lemma 3.5]) that $I(\mathbf{1})$ is an *étale algebra* in $\mathcal{Z}(\mathcal{C})$. Let $\varepsilon\colon I(\mathbf{1}) \to \mathbf{1}$ be a unique map which composes to identity with the unit morphism. Then the morphism $e\colon I(\mathbf{1}) \otimes I(\mathbf{1}) \to I(\mathbf{1}) \xrightarrow{\varepsilon} \mathbf{1}$ is *non-degenerate*, see [DMNO, Remark 3.4]. Thus there is a unique morphism $i\colon \mathbf{1} \to I(\mathbf{1}) \otimes I(\mathbf{1})$ such that the pair (e, i) satisfies the rigidity axiom, see [KO, Definition 1.11]. For any $I(\mathbf{1})$-module M with the action morphism $a\colon I(\mathbf{1}) \otimes M \to M$ consider the following composition (cf. [KO, Fig. 16]):

$$\psi_M\colon M \xrightarrow{i\otimes\mathrm{id}} I(\mathbf{1}) \otimes I(\mathbf{1}) \otimes M \xrightarrow{\mathrm{id}\otimes a} I(\mathbf{1}) \otimes M \xrightarrow{\mathrm{id}\otimes\theta} I(\mathbf{1}) \otimes M \xrightarrow{a} M.$$

Using [KO, Lemma 1.14] one shows that $\mathrm{Tr}(\psi_M) = \dim(I(\mathbf{1}))\,\mathrm{Tr}(\theta_M)$. On the other hand it is proved in [KO, Lemma 4.3] that for a simple M we have $\psi_M = 0$ unless M is *dyslectic*.

The result follows since the simple $I(\mathbf{1})$-modules are precisely $I(X)$, $X \in \mathcal{O}(\mathcal{C})$ (see [DMNO, Lemma 3.5]), and the only simple *dyslectic* module over $I(\mathbf{1})$ is $I(\mathbf{1})$ itself, see [DMNO, Proposition 4.4]. □

Example 2.6. Recall that for a modular category $\mathcal{D}$ its *Gauss sum* is

$$G(\mathcal{D}) = \sum_{A\in\mathcal{O}(\mathcal{D})} \theta_A \dim(A)^2.$$

We can compute the Gauss sum of the category $\mathcal{Z}(\mathcal{C})$ using Theorem 2.5:

$$\begin{aligned}\sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))} \theta_A \dim(A)^2 &= \sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C})),X\in\mathcal{O}(\mathcal{C})} \theta_A \dim(A)[F(A):X]\dim(X)\\ &= \sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C})),X\in\mathcal{O}(\mathcal{C})} \theta_A \dim(A)[I(X):A]\dim(X)\\ &= \sum_{X\in\mathcal{O}(\mathcal{C})} \mathrm{Tr}(\theta_{I(X)})\dim(X) = \dim(\mathcal{C})\end{aligned}$$

Thus we recover the result of Müger [M2, Theorem 1.2] stating that for any spherical $\mathcal{C}$

$$G(\mathcal{Z}(\mathcal{C})) = \dim(\mathcal{C}).$$

Ng and Schauenburg connected $\mathrm{Tr}(\theta^i_{I(X)})$, $i \in \mathbb{Z}$, with *higher Frobenius–Schur indicators*, see [NS]. In particular, $\mathrm{Tr}(\theta^2_{I(X)})$ is related with the classical Frobenius–Schur indicator. Thus we have

Theorem 2.7 [NS, Theorem 4.1]**.** *Assume that $X \in \mathcal{O}(\mathcal{C})$ is self-dual. Then*

$$\mathrm{Tr}(\theta^2_{I(X)}) = \pm \dim(\mathcal{C}). \qquad \square$$

Let $c_{A,B}\colon A \otimes B \to B \otimes A$ denote the braiding on the category $\mathcal{Z}(\mathcal{C})$. For $A, B \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ let $s_{A,B} = \mathrm{Tr}(c_{B,A} \circ c_{A,B})$. We can consider $s_{A,B}$ as a matrix of a linear operator on $K(\mathcal{Z}(\mathcal{C})) \otimes k$ with respect to the basis consisting of classes $[A]$, $A \in \mathcal{O}(\mathcal{C})$. The following result was conjectured by A. Kitaev.

Proposition 2.8 (cf. [KO, Theorem 4.1])**.** *The class $[I(\mathbf{1})] \in K(\mathcal{Z}(\mathcal{C})]$ is an eigenvector for $s = s_{A,B}$ with eigenvalue $\dim(\mathcal{C})$.*

Proof. Recall that $c_{I(\mathbf{1}),A} \circ c_{A,I(\mathbf{1})} = \theta_{A\otimes I(\mathbf{1})}(\theta_A \theta_{I(\mathbf{1})})^{-1}$, see for example [BK, (2.2.8)]. If A is a simple object then θ_A is a scalar; also $\theta_{I(\mathbf{1})} = \mathrm{id}_{I(\mathbf{1})}$ by Theorem 2.5. Thus

$$\mathrm{Tr}(c_{I(\mathbf{1}),A} \circ c_{A,I(\mathbf{1})}) = \theta_A^{-1}\, \mathrm{Tr}(\theta_{A\otimes I(\mathbf{1})}).$$

Observe that $A \otimes I(\mathbf{1})$ is $I(\mathbf{1})$-module, so it is a direct sum of modules $I(X)$, $X \in \mathcal{O}(\mathcal{C})$; in view of Theorem 2.5 the trace above depends only on the multiplicity of $I(\mathbf{1})$ in $A \otimes I(\mathbf{1})$. This multiplicity is

$$\dim_k \mathrm{Hom}_{I(\mathbf{1})}(A \otimes I(\mathbf{1}), I(\mathbf{1})) = \dim_k \mathrm{Hom}(A, I(\mathbf{1})) = [I(\mathbf{1}) : A].$$

Notice that $[I(\mathbf{1}) : A] \neq 0$ implies $\theta_A = 1$ by Theorem 2.5. Thus

$$\begin{aligned} s \cdot [I(\mathbf{1})] &= \sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))} \mathrm{Tr}(c_{I(\mathbf{1}),A} \circ c_{A,I(\mathbf{1})})[A] \\ &= \sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(\mathbf{1}) : A] \dim(\mathcal{C})[A] = \dim(\mathcal{C})[I(\mathbf{1})]. \qquad \square \end{aligned}$$

2.3. Formal codegrees and decomposition of $I(\mathbf{1})$. Let K be a based ring of finite rank and let E be an irreducible representation of K over the field k. Then the element $\alpha_E = \sum_{i\in I} \mathrm{Tr}(b_i, E) b_{i^*} \in K \otimes k$ is in the center of $K \otimes k$; it acts by zero on any irreducible representation of K not isomorphic to E and by a nonzero scalar $f_E \in k$ on E, see [O4, Lemma 2.3]. Let $\mathrm{Irr}(K)$ be the set of isomorphism classes of irreducible representations of K. The scalars f_E, $E \in \mathrm{Irr}(K)$ are called *formal codegrees* of the based ring K or of the fusion category $\mathcal{C}$ with $K = K(\mathcal{C})$.

Example 2.9. Assume that $\mathcal{D}$ is a modular tensor category. It is well known that for any $A \in \mathcal{O}(\mathcal{D})$ the assignment $B \mapsto \frac{s_{A,B}}{\dim(A)}$ is a 1-dimensional representation $\hat{E}_A$ of $K(\mathcal{D})$; moreover the map $A \mapsto \hat{E}_A$ is a bijection $\mathcal{O}(\mathcal{D}) \to \mathrm{Irr}(K(\mathcal{D}))$, see for example [BK, Theorem 3.1.12]. We have (see [O4, Section 3.3])

$$f_{\hat{E}_A} = \frac{\dim(\mathcal{D})}{\dim(A)^2}. \tag{2}$$

Proposition 2.10. *For any based ring K we have*

$$\sum_{E\in \mathrm{Irr}(K)} \frac{\dim(E)}{f_E} = 1.$$

Proof. Use [Lu, Proposition 19.2(c)] with $a = 1$. □

Remark 2.11. It follows from [O4, Lemma 2.6] that the eigenvalues of the operator R (see Proposition 2.2) are $\dim(E)f_E$ appearing with multiplicity $\dim(E)^2$. Thus Proposition 2.10 states that $\mathrm{Tr}(R^{-1}) = 1$.

Remark 2.12. It was observed in [ENO1, proof of Lemma 8.49] that the operator R is positive definite. Thus Remark 2.11 implies that $\dim(E)f_E$ is totally positive for any $E \in \mathrm{Irr}(K(\mathcal{C}))$. Thanks to Proposition 2.10 this can be improved to $f_E \geqslant \dim(E)$. In particular we have $f_E \geqslant 1$ for any representation E.

Let $\mathcal{C}$ be a spherical fusion category and let $E \in \mathrm{Irr}(K(\mathcal{C}))$. Then $K(\mathcal{Z}(\mathcal{C}))$ acts on E via the forgetful homomorphism $K(\mathcal{Z}(\mathcal{C})) \to K(\mathcal{C})$; by Schur's lemma this action factorizes via a character of $K(\mathcal{Z}(\mathcal{C}))$. Let $\hat{E} \in \mathrm{Irr}(K(\mathcal{Z}(\mathcal{C})))$ be the corresponding 1-dimensional representation of $K(\mathcal{Z}(\mathcal{C}))$. Recall (see Section 2.2) that $\mathcal{Z}(\mathcal{C})$ is a modular tensor category with $\dim(\mathcal{Z}(\mathcal{C})) = \dim(\mathcal{C})^2$. Thus by Example 2.9 there exists a unique $A_E \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ such that $\hat{E} = \hat{E}_{A_E}$; in addition

$$f_{\hat{E}} = \frac{\dim(\mathcal{C})^2}{\dim(A_E)^2}. \tag{3}$$

Theorem 2.13. *For a spherical fusion category $\mathcal{C}$ the assignment $E \mapsto A_E$ is an embedding $\mathrm{Irr}(K(\mathcal{C})) \subset \mathcal{O}(\mathcal{Z}(\mathcal{C}))$. Its image consists of $A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ with $[I(\mathbf{1}) : A] \neq 0$. Moreover, we have*

$$[I(\mathbf{1}) : A_E] = \dim(E), \quad \textit{and} \quad \dim A_E = \frac{\dim(\mathcal{C})}{f_E}. \tag{4}$$

Proof. It is known that $K(\mathcal{Z}(\mathcal{C})) \otimes k$ maps surjectively on the center of $K(\mathcal{C}) \otimes k$, see [ENO1, Lemma 8.49]; the first statement follows. Also [O4, Lemma 3.1] states that $f_{\hat{E}} = f_E^2$, so (3) implies $f_E = \pm\frac{\dim(\mathcal{C})}{\dim(A_E)}$.

In view of Proposition 2.8 and Example 2.9 we have

$$[I(\mathbf{1})] = \sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(\mathbf{1}) : A]\frac{\dim(A)}{\dim(\mathcal{C})}\alpha_{\hat{E}_A}.$$

Thus $[I(\mathbf{1})]$ acts on $E \in \mathrm{Irr}(K(\mathcal{C}))$ by the scalar

$$[I(\mathbf{1}) : A_E]\frac{\dim(A_E)}{\dim(\mathcal{C})} f_{\hat{E}_{A_E}} = [I(\mathbf{1}) : A_E]\frac{\dim(\mathcal{C})}{\dim(A_E)},$$

see (3). On the other hand by Proposition 2.4 we have $[F(I(\mathbf{1})] = R$, so by [O4, Lemma 2.6] it acts on E by the scalar $\dim(E)f_E$. Hence

$$\dim(E)f_E = [I(\mathbf{1}) : A_E]\frac{\dim(\mathcal{C})}{\dim(A_E)}.$$

Combining this with $f_E = \pm\frac{\dim(\mathcal{C})}{\dim(A_E)}$ we get (4).

It is known (see [ENO1, Proposition 5.6]) that $\sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))}[I(\mathbf{1}):A]^2$ equals the rank of $\mathcal{C}$. We already proved that each object A_E appears in $I(\mathbf{1})$ with multiplicity $\dim(E)$. Thus

$$\sum_{E\in\mathrm{Irr}(K(\mathcal{C}))}[I(\mathbf{1}):A_E]^2=\sum_{E\in\mathrm{Irr}(K(\mathcal{C}))}\dim(E)^2=\dim(K(\mathcal{C})\otimes k)=\sum_{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))}[I(\mathbf{1}):A]^2.$$

It follows that any object $A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))$ with $[I(\mathbf{1}):A]\neq 0$ is of the form A_E for some $E\in\mathrm{Irr}(K(\mathcal{C}))$. The theorem is proved. □

Corollary 2.14 (cf. [ENO1, Proposition 8.22])**.** *For a not necessary spherical fusion category $\mathcal{C}$ the formal codegrees f_E divide* $\dim(\mathcal{C})$.

Proof. In a case of spherical $\mathcal{C}$ the result is immediate from Theorem 2.13 since $\dim(A_E)$ is an algebraic integer. For a general fusion category $\mathcal{C}$ one considers its *sphericalization* $\tilde{\mathcal{C}}$, see [ENO1, Remark 3.1] or Section 5.1 below. This is spherical category of global dimension $2\dim(\mathcal{C})$; moreover $2f_E$ is a formal codegree of $\tilde{\mathcal{C}}$ for any $E\in\mathrm{Irr}(K(\mathcal{C}))$. The result follows. □

Assume again that $\mathcal{C}$ is spherical. Let $L(\mathcal{C})$ be the *dimension field* of $\mathcal{C}$, that is the field generated over $\mathbb{Q}$ by the dimensions of simple objects of $\mathcal{C}$.

Corollary 2.15. *The formal codegrees of $\mathcal{C}$ lie in the field $L(\mathcal{C})$.*

Proof. For any $A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))$ we have $\dim(A)=\dim(F(A))\in K$. Since $\dim(\mathcal{C})\in L(\mathcal{C})$ the result follows from (4). □

Corollary 2.16. *There is an isomorphism of associative algebras*

$$K(\mathcal{C})\otimes k\simeq \mathrm{End}(I(\mathbf{1})).$$

Proof. Both algebras are isomorphic to direct sum of matrix algebras of sizes $\dim(E)$ indexed by $E\in\mathrm{Irr}(K(\mathcal{C}))$. □

We don't know whether there is a canonical choice of the isomorphism in Corollary 2.16.

Example 2.17. Assume that $\mathcal{C}$ is dual to a modular tensor category $\mathcal{D}$. Then $\mathcal{Z}(\mathcal{C})=\mathcal{D}\boxtimes\mathcal{D}^{\mathrm{rev}}$ (see [M2, Theorem 7.10]) and $[I(\mathbf{1})]\in K(\mathcal{Z}(\mathcal{C}))=K(\mathcal{D})\otimes K(\mathcal{D})$ is known as *modular invariant* associated with $\mathcal{C}$ (or rather with module category $\mathcal{M}$ over $\mathcal{D}$ such that $\mathcal{C}=\mathcal{D}^*_{\mathcal{M}}$), see [BEK, Definition 5.5], [FRS1, Section 5.3]. In this case our results are well known: Theorem 2.5 says that the modular invariant commutes with T-matrix (cf. [BEK, Theorem 5.7], [FRS1, Theorem 5.1]), Proposition 2.8 says that modular invariant commutes with S-matrix (cf. [BEK, Theorem 5.7], [FRS1, Theorem 5.1]), and Corollary 2.16 expresses $K(\mathcal{C})\otimes k$ in terms of the modular invariant (cf. [BEK, Theorem 6.8], [FRS2, Theorem O]).

Example 2.18 (joint with D. Nikshych)**.** It follows immediately from Corollary 2.16 that $K(\mathcal{C})$ is commutative if and only if $I(\mathbf{1})$ is multiplicity free. Let us consider minimal étale subalgebras of $I(\mathbf{1})$. Such subalgebras intersect trivially, so no nontrivial simple summand of $I(\mathbf{1})$ appears in two such subalgebras. Thus the number of such subalgebras is at most $\mathrm{rk}(\mathcal{C})-1$. In view of [DMNO, Theorem

4.10] this translates into the following statement: the number of maximal fusion subcategories of $\mathcal{C}$ is at most $\mathrm{rk}(\mathcal{C})-1$. It was conjectured in [GX, Conjecture 1.2] that this statement holds for any fusion category $\mathcal{C}$ (possibly with non-commutative $K(\mathcal{C})$). However recent counterexamples to Wall's conjecture (see [GHPS]) show that it fails even for some pointed fusion categories. We don't know whether this statement fails for a commutative unital based ring K of finite rank.

Remark 2.19. Let $\mathcal{C}=\bigoplus_{i,j}\mathcal{C}_{ij}$ be an indecomposable multi-fusion category (see [ENO1, Section 2.4]) with decomposable unit object $\mathbf{1}=\bigoplus_i \mathbf{1}_i$. Assume that $\mathcal{Z}(\mathcal{C}_1)$ is spherical and $\theta_{I(\mathbf{1})}=\mathrm{id}_{I(\mathbf{1})}$. The proof of Theorem 2.13 and Corollary 2.16 extends with trivial changes to this setting. Moreover, Theorem 2.13 applied to fusion category $\mathcal{C}_{ii}$ implies that the dimension of $[\mathbf{1}_i]E$ equals $[I(\mathbf{1}_i):A_E]$. It follows that we can choose an isomorphism in Corollary 2.16 sending $[\mathbf{1}_i]\in K(\mathcal{C}_1)$ to the projection of $I(\mathbf{1})$ onto $I(\mathbf{1}_i)$. This implies that the $(K(\mathcal{C}_{ii})\otimes k,\, K(\mathcal{C}_{jj})\otimes k)$-bimodule $K(\mathcal{C}_{ij})\otimes k$ can be identified with the $(\mathrm{End}(I(\mathbf{1}_i)),\, \mathrm{End}(I(\mathbf{1}_j)))$-bimodule $\mathrm{Hom}(I(\mathbf{1}_i),\, I(\mathbf{1}_j))$, where $K(\mathcal{C}_{ii})\otimes k$ and $K(\mathcal{C}_{jj})\otimes k$ are identified with $\mathrm{End}(I(\mathbf{1}_i))$ and $\mathrm{End}(I(\mathbf{1}_j))$ as in Corollary 2.16. In particular, the number of simple objects in $\mathcal{C}_{ij}$ equals $\dim \mathrm{Hom}(I(\mathbf{1}_i),\, I(\mathbf{1}_j))$.

For example this applies as follows: let $\mathcal{D}$ be a spherical fusion category and let $\mathcal{M},\mathcal{N}$ be indecomposable semisimple module categories over $\mathcal{D}$. Let $M,\, N$ be Lagrangian subalgebras in $\mathcal{Z}(\mathcal{D})$ associated with $\mathcal{M}$ and $\mathcal{N}$ in [DMNO, Section 4.2] (so $M=I_{\mathcal{M}}(\mathbf{1}_{\mathcal{M}})$, where $I_{\mathcal{M}}$ is the adjoint of the canonical functor $\mathcal{Z}(\mathcal{D})\to\mathcal{D}^*_{\mathcal{M}}$ to the dual of $\mathcal{D}$ with respect to $\mathcal{M}$, and similarly for N). Assume that $\theta_M=\mathrm{id}_M$ and $\theta_N=\mathrm{id}_N$. Then applying the above statement to $\mathcal{C}=\mathcal{D}^*_{\mathcal{M}\oplus\mathcal{N}}$ we get the following: the $(K(\mathcal{D}^*_{\mathcal{M}})\otimes k,\, K(\mathcal{D}^*_{\mathcal{N}})\otimes k)$-bimodule $K(\mathrm{Fun}_{\mathcal{D}}(\mathcal{M},\,\mathcal{N}))\otimes k$ is isomorphic to the $(\mathrm{End}(M),\,\mathrm{End}(N))$-bimodule $\mathrm{Hom}(M,\,N)$, where $K(\mathcal{D}^*_{\mathcal{M}})\otimes k$ is identified with $\mathrm{End}(M)$ as in Corollary 2.16 and similarly for $K(\mathcal{D}^*_{\mathcal{N}})$. In particular the number of simple objects in $\mathrm{Fun}_{\mathcal{D}}(\mathcal{M},\,\mathcal{N})$ equals $\dim\mathrm{Hom}(M,\,N)$. This statement can be considered as a generalization of the theory of quantum Manin triples from [DMNO, Section 4.4]. In the case of modular category $\mathcal{D}$ this result was obtained in [BEK, Theorem 6.12] and [FRS1, Theorem 5.18].

Note also that the condition $\theta_M=\mathrm{id}_M$ (and similarly for N) is satisfied automatically when the dimensions in the category $\mathcal{Z}(\mathcal{D})$ are positive. In this case the category $\mathcal{D}^*_{\mathcal{M}}$ is pseudo-unitary. Indeed, it is known that $\mathcal{Z}(\mathcal{D}^*_{\mathcal{M}})\simeq\mathcal{Z}(\mathcal{D})$ (see [M1, Remark 3.18], [O1, Corollary 2.6]), so the result follows from [ENO1, Theorem 2.15 and Proposition 8.12]. Thus the spherical structure on $\mathcal{Z}(\mathcal{D})=\mathcal{Z}(\mathcal{D}^*_{\mathcal{M}})$ is the same as the one obtained from a spherical structure on $\mathcal{D}^*_{\mathcal{M}}$ (defined in [ENO1, Proposition 8.23]) and one deduces $\theta_M=\mathrm{id}_M$ from Theorem 2.5. On the other hand the condition $\theta_M=\mathrm{id}_M$ is not always satisfied: it fails for $\mathcal{D}=\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ (see [ENO1, Example 1 in Section 2]) with non-standard spherical structure (so the dimension of the nontrivial object is -1) and module category $\mathcal{M}$ with one simple object. Thus it would be interesting to investigate whether the conclusions above hold true in the case when $\theta_M=\mathrm{id}_M$ fails (this is the case in the example above since we can replace the non-standard spherical structure by the standard one).

Example 2.20. Let $\mathcal{C}$ be a spherical fusion category and let $I(\mathbf{1})\in\mathcal{Z}(\mathcal{C})$ be as above. It was asked by A. Kitaev whether there exists an étale algebra $\tilde{I}$ such that $\tilde{I} = I(\mathbf{1})$ as an object of $\mathcal{Z}(\mathcal{C})$ but the étale algebras $\tilde{I}$ and $I(\mathbf{1})$ are not isomorphic. In view of Remark 2.19 this translates into the following question: is there a module category $\mathcal{M}$ over $\mathcal{C}$ such that the $K(\mathcal{C})$-module $K(\mathcal{M}) \otimes k$ is isomorphic to $K(\mathcal{C}) \otimes k$? Such examples do exist: for example let $\mathcal{V}$ be $\mathbb{Z}/2\mathbb{Z}$-graded fusion category constructed in [CMS, Theorem A.5.1]; then $\mathcal{V}_1$ considered as a module category over $\mathcal{C} = \mathcal{V}_0$ satisfies the condition above. Thus the question above has a positive answer.

2.4. Pseudo-unitary inequality. Let us assume that the category $\mathcal{C}$ is *pseudo-unitary*, see [ENO1, Section 8.4]. Thus the category $\mathcal{C}$ has a spherical structure such that the dimensions of objects are nonnegative.

Theorem 2.21. *Let $\mathcal{C}$ be a pseudo-unitary fusion category. Then the formal codegrees of $\mathcal{C}$ satisfy*

$$\sum_{E \in \mathrm{Irr}(K(\mathcal{C}))} \frac{1}{f_E^2} \leqslant \frac{1}{2}\left(1 + \frac{1}{\dim(\mathcal{C})}\right). \tag{5}$$

Remark 2.22. It follows from Remark 2.11 that the LHS of (5) equals $\mathrm{Tr}(R^{-2})$. Since $\dim(\mathcal{C})$ is the largest eigenvalue of R, the inequality (5) can be stated purely in terms of operator R.

Proof. Let $X \in \mathcal{O}(\mathcal{C}) \setminus \{\mathbf{1}\}$. By Theorem 2.5

$$\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(X) : A]\theta_A \dim(A) = \mathrm{Tr}(\theta_{I(X)}) = 0,$$

and $\theta_A = 1$ for any A appearing in $I(\mathbf{1})$. Thus

$$\sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] \neq 0}} [I(X) : A] \dim(A) = - \sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] = 0}} [I(X) : A]\theta_A \dim(A).$$

Since all dimensions are positive real numbers we have

$$\begin{aligned}\sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] \neq 0}} [I(X) : A] \dim(A) &= \left| \sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] \neq 0}} [I(X) : A] \dim(A) \right| \\ &= \left| \sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] = 0}} [I(X) : A]\theta_A \dim(A) \right| \leqslant \sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] = 0}} [I(X) : A] \dim(A).\end{aligned}$$

Therefore

$$\sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] \neq 0}} [I(X) : A] \dim(A) \leqslant \frac{1}{2} \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(X) : A] \dim(A) = \frac{1}{2} \dim(I(X)).$$

Recall that $\dim(I(X)) = \dim(X)\dim(\mathcal{C})$, see (1). Hence multiplying the inequality above by $\dim(X)$ and summing over $X \in \mathcal{O}(\mathcal{C}) \setminus \{\mathbf{1}\}$ we get

$$\sum_{\mathbf{1} \neq X \in \mathcal{O}(\mathcal{C})} \sum_{\substack{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \\ [I(\mathbf{1}):A] \neq 0}} [I(X) : A] \dim(X) \dim(A) \leqslant \sum_{\mathbf{1} \neq X \in \mathcal{O}(\mathcal{C})} \frac{1}{2} \dim(X)^2 \dim(\mathcal{C}).$$

Observe that

$$\sum_{\mathbf{1}\neq X\in\mathcal{O}(\mathcal{C})}[I(X):A]\dim(X)=\sum_{\mathbf{1}\neq X\in\mathcal{O}(\mathcal{C})}[F(A):X]\dim(X)=\dim(A)-[F(A):\mathbf{1}],$$

and

$$\sum_{\mathbf{1}\neq X\in\mathcal{O}(\mathcal{C})}\dim(X)^2=\dim(\mathcal{C})-1.$$

Therefore we get

$$\sum_{\substack{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))\\ [F(A):\mathbf{1}]\neq 0}}(\dim(A)-[F(A):\mathbf{1}])\dim(A)\leqslant\frac{1}{2}(\dim(\mathcal{C})-1)\dim(\mathcal{C}).\tag{6}$$

Using (1) we compute

$$\sum_{\substack{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))\\ [I(\mathbf{1}):A]\neq 0}}[F(A):\mathbf{1}]\dim(A)=\sum_{\substack{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))\\ [I(\mathbf{1}):A]\neq 0}}[I(\mathbf{1}):A]\dim(A)=\dim(I(\mathbf{1}))=\dim(\mathcal{C}).$$

Thus (6) is equivalent to

$$\left(\sum_{\substack{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))\\ [I(\mathbf{1}):A]\neq 0}}\dim(A)^2\right)-\dim(\mathcal{C})\leqslant\frac{1}{2}(\dim(\mathcal{C})-1)\dim(\mathcal{C}),$$

or

$$\sum_{\substack{A\in\mathcal{O}(\mathcal{Z}(\mathcal{C}))\\ [I(\mathbf{1}):A]\neq 0}}\dim(A)^2\leqslant\frac{1}{2}(\dim(\mathcal{C})+1)\dim(\mathcal{C}).$$

By Theorem 2.13 this inequality is equivalent to (5). □

3. Categorifiable Simple Based Rings of Rank 3

3.1. Based rings of rank 3. Let k, l, m, n be integers satisfying the equality

$$k^2+l^2=kn+lm+1.\tag{7}$$

It is easy to check that a ring $K(k,l,m,n)$ with basis 1, X, Y and multiplication

$$X^2=1+mX+kY,\quad Y^2=1+lX+nY,\quad XY=YX=kX+lY\tag{8}$$

is associative. In the case where the numbers k, l, m, n are nonnegative the ring $K(k,l,m,n)$ is a based ring.

Another based ring of rank 3 is the group algebra of the group $\mathbb{Z}/3\mathbb{Z}$ with basis given by the group elements which we denote as $K(\mathbb{Z}/3\mathbb{Z})$.

We have the following

Proposition 3.1 (cf. [O3, Section 3.1])**.** *Any unital based ring of rank* 3 *is isomorphic to either* $K(k,l,m,n)$ *or* $K(\mathbb{Z}/3\mathbb{Z})$.

Proof. In the case when any element of the based ring is self-dual (8) is a general form of fusion rules and (7) is equivalent to the associativity of multiplication (8).

Assume that not every element of the based ring is self-dual. Then we have a basis 1, X, Y with $X^* = Y$ whence $\mathrm{FPdim}(X) = \mathrm{FPdim}(Y)$. The element X^2 is an integral linear combination of X and Y, hence $\mathrm{FPdim}(X) \in \mathbb{Z}$. The element XY is 1 plus an integral linear combination of X and Y, so $\mathrm{FPdim}(X)$ is a divisor of 1. Thus $\mathrm{FPdim}(X) = \mathrm{FPdim}(Y) = 1$ and the proposition is proved. □

Remark 3.2. (i) Notice that the interchange $X \leftrightarrow Y$ induces a based ring isomorphism $K(k, l, m, n) \simeq K(l, k, n, m)$. Thus we will often assume that $l \leqslant k$.

(ii) The diophantine equation (7) has infinitely many solutions. Clearly, for any such solution k and l are coprime. The number of (nonnegative) solutions with $0 < l \leqslant k \leqslant x$ is about $Cx^2 \ln(x)$ for a constant $C \approx \frac{1}{3}$ (notice that there are infinitely many solutions with $k = 1$, $l = 0$).

The main result of this section is the following

Theorem 3.3. *Let $\mathcal{C}$ be a pivotal fusion category of rank 3. Then $K(\mathcal{C})$ is isomorphic to one of the following*:
$K(1, 0, m, 0)$ *for some nonnegative integer m*;
$K(1, 1, 1, 0)$;
$K(\mathbb{Z}/3\mathbb{Z})$.

By Proposition 3.1 we can and will assume that $K(\mathcal{C}) = K(k, l, m, n)$ in the proof of Theorem 3.3.

Proof of Theorem 3.3. Combine Lemma 3.4, Propositions 3.5 and 3.8 with Lemmas 3.7 and 3.10 below. □

3.2. Notations. The following notation will be used throughout the paper. We consider an element $1+X^2+Y^2 = 3+(l+m)X+(k+n)Y \in K(k, l, m, n)$ and the operator $R\colon K(k, l, m, n) \to K(k, l, m, n)$ of left multiplication by this element. We have explicitly

$$R = \begin{pmatrix} 3 & l+m & k+n \\ l+m & 3+(l+m)m+(k+n)k & (l+m)k+(k+n)l \\ k+n & (l+m)k+(k+n)l & 3+(l+m)l+(k+n)n \end{pmatrix} \tag{9}$$

In what follows $p(t) = \det(t-R)$ denote the characteristic polynomial of R. The roots of the polynomial $p(t)$ are real positive (since the matrix above is symmetric positive definite); we will denote the roots by f_1, f_2, f_3 assuming that $f_1 \leqslant f_2 \leqslant f_3$. It is clear that the numbers f_1, f_2, f_3 are precisely the formal codegrees of $K(k, l, m, n)$. Thus Proposition 2.10 implies that $\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3} = 1$, or, equivalently, $p(t) = t^3 - bt^2 + ct - c$ for some positive integers $b = \mathrm{Tr}(R)$ and $c = \det(R)$. Using (9) we have

$$b = (k+n)^2 + (l+m)^2 + 9 \tag{10}$$

The formula for c is more complicated, see (12) below.

3.3. Pivotal categories of rank 3 are pseudo-unitary

Lemma 3.4. *Let $\mathcal{C}$ be a pivotal fusion category of rank 3. Then some Galois conjugate of $\mathcal{C}$ is pseudo-unitary.*

Proof. Since $K(\mathcal{C})$ is a commutative semisimple ring there are 3 distinct homomorphisms $\phi_i\colon K(\mathcal{C}) \to \mathbb{C}$, $i = 1, 2, 3$. One of this homomorphisms is the categorical dimension and another one is the Frobenius–Perron dimension. We would like to show that these two homomorphisms are in the same orbit under the action of the Galois group. For a sake of contradiction assume that this is not the case. Then the orbit of one of them has size 1, that is one of these homomorphisms lands into $\mathbb{Z}$. If the Frobenius–Perron dimension lands in $\mathbb{Z}$ then the category $\mathcal{C}$ is integral and we are done by [ENO1, Proposition 8.24]. If the categorical dimension lands in $\mathbb{Z}$ the category $\mathcal{C}$ is pseudo-unitary by [HR, Lemma A.1]. □

We will assume from now on that $\mathcal{C}$ is pseudo-unitary. In particular $\dim(\mathcal{C}) = f_3$.

3.4. The generic case. The pseudo-unitary inequality (5) applied to $K(k, l, m, n)$ says:

$$\frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} \leqslant \frac{1}{2}\left(1 + \frac{1}{f_3}\right).$$

Since

$$\frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} = \left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right)^2 - 2\frac{f_1 + f_2 + f_3}{f_1 f_2 f_3} = 1 - 2\frac{b}{c},$$

we get an equivalent inequality

$$1 - \frac{4b}{c} \leqslant \frac{1}{f_3}.$$

If $c - 4b > 0$ this is equivalent to

$$f_3 \leqslant \frac{c}{c - 4b}. \qquad (11)$$

Proposition 3.5. *Assume that $c - 4b > 4$ and $b \geqslant 50$. Then there is no pseudo-unitary fusion category $\mathcal{C}$ such that $K(\mathcal{C}) = K(k, l, m, n)$.*

Proof. The Vieta theorem implies that $f_3 \geqslant \frac{1}{3}b > 9$. Under the assumptions of the proposition $c > 4b$, so (11) applies. We get

$$\frac{c}{c - 4b} > 9 \;\Leftrightarrow\; c < \frac{9}{2}b.$$

Lemma 3.6. *Under the assumptions of the proposition, we have*

$$1 < f_1 < 2 < f_2 < b - 5 < f_3 < b.$$

Proof. We have $p(1) = 1 - b < 0$, $p(2) = 8 + c - 4b > 0$ and $p(b) = bc - c > 0$. Also

$$p(b-5) = (b-5)((b-5)^2 - b(b-5) + c) - c = (b-5)\left(c - \frac{9}{2}b + \frac{1}{2}(50 - b)\right) - c < 0.$$

The lemma follows. □

By the assumptions of the proposition we have $c - 4b \geqslant 5$. Thus

$$\frac{c}{5} \geqslant \frac{c}{c-4b} \geqslant f_3 > b - 5,$$

whence

$$c > 5b - 25 \;\Leftrightarrow\; \left(c - \frac{9}{2}b\right) + \frac{1}{2}(50 - b) > 0.$$

We get a contradiction with inequalities $b \geqslant 50$ and $c < \frac{9}{2}b$, which proves the proposition. □

3.5. Small cases. In this section we consider the cases not covered by Proposition 3.5. Thus either $b < 50$ or $c - 4b \leqslant 4$, or both.

3.5.1. *The case* $b < 50$.

Lemma 3.7. *Let* $\mathcal{C}$ *be a fusion category such that* $K(\mathcal{C}) = K(k, l, m, n)$ *with* $b < 50$. *Then* $K(\mathcal{C})$ *is isomorphic to one of the following*: $K(1, 0, m, 0)$ *with* $m \leqslant 6$, $K(1, 1, 1, 0)$, $K(2, 1, 2, 1)$.

Proof. Using (10) and (7) we get

$$b = 3k^2 + 3l^2 + n^2 + m^2 + 7.$$

Assuming $k \geqslant l > 0$ we get from inequality $b < 50$ that $k \leqslant 3$. There are 11 based rings $K(k, l, m, n)$ with $3 \geqslant k \geqslant l > 0$ but out of those only $K(1, 1, 1, 0)$ and $K(2, 1, 2, 1)$ pass both the cyclotomic and d-number tests.

If $l = 0$ then $k = 1$ since k and l are coprime. By (7) this implies that $n = 0$ and $b = 10 + m^2$. The inequality $b < 50$ gives $m \leqslant 6$ and we are done. □

3.5.2. *The case* $c - 4b \leqslant 4$.

Proposition 3.8. *Let* $\mathcal{C}$ *be a fusion category such that* $K(\mathcal{C}) = K(k, l, m, n)$ *with* $c - 4b \leqslant 4$. *Then* $K(\mathcal{C})$ *is isomorphic to one of the following*: $K(1, 0, m, 0)$ *with* $m \in \mathbb{Z}_{\geqslant 0}$, $K(1, 1, 1, 0)$, $K(2, 1, 2, 1)$.

Proof. We have the following formula verified by a direct calculation using (9):

$$p(2) = 8 + c - 4b = (kl - mn)^2 + (k^2 + kn - l^2 - ml)^2 - 1. \tag{12}$$

Hence the inequality $c - 4b \leqslant 4$ is equivalent to

$$(kl - mn)^2 + (k^2 + kn - l^2 - ml)^2 \leqslant 13. \tag{13}$$

Observe that in view of (7) the number $k^2 + kn - l^2 - lm = 2(k^2 - lm) - 1$ is odd. By Remark 3.2 (i) we can assume that this number is positive. Then (13) implies that $k^2 - lm = 1$ or $k^2 - lm = 2$. Now we consider these cases separately.

Case $k^2 - lm = 1$. In this case $l^2 - kn = 0$ by (7). Since k and l are coprime this implies $k = 1$, $l^2 = n$, $lm = 0$. In the case $l = 0$ we get the based ring $K(1, 0, m, 0)$. Otherwise, $m = 0$ and (13) says

$$l^2 + 1 \leqslant 13 \;\Leftrightarrow\; l \leqslant 3.$$

The rings $K(1, 3, 0, 9)$ (with $b = 118$, $c = 473$) and $K(1, 2, 0, 4)$ (with $b = 38$, $c = 148$) do not pass neither cyclotomic nor d-number tests. Thus the only possibility is $K(1, 1, 0, 1) \simeq K(1, 1, 1, 0)$ and the conclusion of the proposition holds in this case.

Case $k^2 - lm = 2$. In this case $l^2 - kn = -1$ whence $m = \frac{k^2-2}{l}$ and $n = \frac{l^2+1}{k}$. We compute

$$kl - mn = \frac{2l^2 - k^2 + 2}{kl}.$$

The inequality (13) says $(kl - mn)^2 + 9 \leqslant 13$, so we have the following subcases:

Subcase $kl-mn = \pm 2$. Equivalently, $2l^2-k^2+2 = \pm 2kl$ implying $3l^2+2 = (k\pm l)^2$ and we get a contradiction considering this equality modulo 3.

Subcase $kl-mn = \pm 1$. Equivalently, $2l^2-k^2+2 = \pm kl$ implying $9l^2+8 = (2k\pm l)^2$ and again we get a contradiction considering this equality modulo 3.

Subcase $kl - mn = 0$. Equivalently, $2l^2 - k^2 + 2 = 0$. This diophantine equation has infinitely many solutions. We compute $m = 2l$, $n = \frac{k}{2}$, $b = 27\frac{l^2+1}{2}$. Equation (12) gives $c = 4b$. The polynomial $p(t) = t^3 - bt^2 + 4bt - 4b$ satisfies $p(2) = 8$. Hence if γ is an integer root of $p(t)$ then $2 - \gamma$ is a divisor of 8 and one verifies easily that $p(t)$ is irreducible for $b > 27$. Observe that $b = 27\frac{l^2+1}{2}$ is odd, so $\frac{b^3}{c} = \frac{b^2}{4} \notin \mathbb{Z}$. Thus for $b > 27$ the d-number test is failed (one can also show that the cyclotomic test fails as well). Finally, $27\frac{l^2+1}{2} = b \leqslant 27$ implies $l = 1$ and we get the based ring $K(2, 1, 2, 1)$, so the conclusion of the proposition holds.

The proposition is proved. □

Remark 3.9. The crucial role in the proof above is played by the identity (12). Thus it seems to be of great interest to find its counterparts for based rings of higher ranks.

3.6. The ring $K(2, 1, 2, 1)$

Lemma 3.10. *There exists no pseudo-unitary fusion category $\mathcal{C}$ for which $K(\mathcal{C}) = K(2, 1, 2, 1)$.*

Proof. We have $p(t) = t^3 - 27t^2 + 108t - 108$ in this case, so $f_1 = 12 - 6\sqrt{3}$, $f_2 = 3$, $f_3 = 12+6\sqrt{3}$. Using Theorem 2.13 we get that $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ decomposes into simple objects as $\mathbf{1} \oplus A \oplus B$, where $F(A) = \mathbf{1} \oplus X \oplus Y$ and $F(B) = \mathbf{1} \oplus 2X \oplus 2Y$. Using Proposition 2.4 we see that $I(Y)$ decomposes as $A + 2B + S$, where $F(A + 2B) = 3 + 5X + 5Y$ and S is a (non simple) object with $F(S) = 4X + 4Y$. This implies that $\mathrm{Tr}(\theta_{I(Y)}) \neq 0$ and we have a contradiction with Theorem 2.5. □

Remark 3.11. We will show later (see Theorem 5.1) that a categorification of $K(2, 1, 2, 1)$ admits a spherical (and hence pivotal) structure. Thus by the lemma above and Lemma 3.4 the ring $K(2, 1, 2, 1)$ is not categorifiable.

4. Non-Simple Based Rings of Rank 3

4.1. The main result of this section is the following

Theorem 4.1. *Assume that the based ring $K(1, 0, m, 0)$ is categorifiable. Then $m \leqslant 2$.*

The multiplication in the based ring $K(1, 0, m, 0)$ is given by

$$Y^2 = 1, \quad XY = YX = X, \quad X^2 = 1 + mX + Y. \tag{14}$$

Thus, any category with $K(\mathcal{C}) = K(1, 0, m, 0)$ is an example of *near-group category* associated with finite group $\mathbb{Z}/2\mathbb{Z}$, see for example [T, Definition 1.1]. In particular, such category or its Galois conjugate is pseudo-unitary by [T, Theorem 1.3].

In order to prove Theorem 4.1 we will assume for the sake of contradiction that $m > 2$ and $\mathcal{C}$ is a pseudo-unitary fusion category with $K(\mathcal{C}) = K(1, 0, m, 0)$. We will endow $\mathcal{C}$ with the pseudo-unitary spherical structure. We set $d = \dim(X) = \mathrm{FPdim}(X)$.

Lemma 4.2. *We have $d = \frac{m}{2} + \sqrt{\frac{m^2}{4} + 2}$. In particular $d \notin \mathbb{Z}$ for $m \geqslant 2$.* $\square$

4.2. Simple objects in $\mathcal{Z}(\mathcal{C})$. One of the formal codegrees of the based ring $K(1, 0, m, 0)$ is $f_1 = 2$. By Theorem 2.13 the object $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ decomposes into the sum of 3 simple objects $I(\mathbf{1}) = \mathbf{1} \oplus A \oplus B$; we can assume that $\mathrm{FPdim}(B) = \frac{\mathrm{FPdim}(\mathcal{C})}{2}$ whence $\mathrm{FPdim}(A) = \frac{\mathrm{FPdim}(\mathcal{C})}{2} - 1$. Using Proposition 2.4 we deduce that

$$F(A) = \mathbf{1} \oplus \frac{m}{2}X, \quad F(B) = \mathbf{1} \oplus Y \oplus \frac{m}{2}X. \tag{15}$$

In particular, m is forced to be even. Theorem 2.5 states that $\theta_A = \theta_B = 1$.

Next we consider $I(Y)$. By Proposition 2.4 we have $\dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(Y), I(Y)) = 3$ and $\dim \mathrm{Hom}(I(Y), I(\mathbf{1})) = 1$; we also have by definition $\dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(Y), B) = 1$. Thus we have a decomposition $I(Y) = B + C + D$, where C and D are distinct simple objects different from A and B. Using Proposition 2.4 we have

$$F(C) = Y + \alpha X, \quad F(D) = Y + \beta X, \tag{16}$$

where $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$ and $\alpha + \beta = \frac{m}{2}$. It is clear from Theorem 2.5 that $\theta_C = \theta_D = -1$.

Finally consider $I(X)$. We have

$$\dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), A) = \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), B) = \frac{m}{2},$$
$$\dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), C) = \alpha, \quad \dim \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), D) = \beta.$$

Thus

$$I(X) = \frac{m}{2}A + \frac{m}{2}B + \alpha C + \beta D + \sum_i \gamma_i E_i, \tag{17}$$

where $\gamma_i \in \mathbb{Z}_{>0}$ and the objects $E_i \in \mathcal{Z}(\mathcal{C})$ satisfy $F(E_i) = \gamma_i X$. Using Proposition 2.4 we compute

$$\frac{m^2}{4} + \frac{m^2}{4} + \alpha^2 + \beta^2 + \sum_i \gamma_i^2 = \dim \mathrm{Hom}(F(I(Y), Y) = m^2 + 4$$

whence

$$\sum_i \gamma_i^2 = \frac{m^2}{2} + 4 - \alpha^2 - \beta^2 = \frac{m^2}{4} + 4 + 2\alpha\beta. \tag{18}$$

Set $\theta_i = \theta_{E_i}$. Theorem 2.5 implies

$$\frac{m}{2}\left(1+\frac{m}{2}d\right)+\frac{m}{2}\left(2+\frac{m}{2}d\right)-\alpha(1+\alpha d)-\beta(1+\beta d)+\sum_i \gamma_i^2\theta_i d=0$$

whence

$$\sum_i \gamma_i^2\theta_i = \alpha^2+\beta^2-\frac{m^2}{2}-\frac{m}{d} = -2\alpha\beta-\frac{m}{2}\sqrt{\frac{m^2}{4}+2}. \tag{19}$$

Similarly, Theorem 2.7 implies

$$\frac{m}{2}\left(1+\frac{m}{2}d\right)+\frac{m}{2}\left(2+\frac{m}{2}d\right)+\alpha(1+\alpha d)+\beta(1+\beta d)+\sum_i \gamma_i^2\theta_i^2 d$$
$$=\pm\dim(\mathcal{C})=\pm(4+md)$$

whence

$$\sum_i \gamma_i^2\theta_i^2 = 2\alpha\beta-\frac{m^2}{4}-(m\mp 2)\sqrt{\frac{m^2}{4}+2}. \tag{20}$$

4.3. Proof of Theorem 4.1. Assume that $m>2$ and the based ring $K(1,0,m,0)$ is categorifiable. Recall that this implies that m is even. Combining equations (18) and (20) we obtain

$$\frac{m^2}{4}+4+2\alpha\beta=\sum_i \gamma_i^2 \geqslant \left|\sum_i \gamma_i^2\theta_i^2\right| = \frac{m^2}{4}+(m\mp 2)\sqrt{\frac{m^2}{4}+2}-2\alpha\beta.$$

Equivalently, $4\alpha\beta+4\geqslant (m\mp 2)\sqrt{\frac{m^2}{4}+2}$, which implies

$$\frac{m^2}{4}+4\geqslant 4\alpha\beta+4\geqslant (m\mp 2)\sqrt{\frac{m^2}{4}+2}\geqslant (m-2)\sqrt{\frac{m^2}{4}+2}.$$

It is easy to see that the inequality $\frac{m^2}{4}+4\geqslant (m-2)\sqrt{\frac{m^2}{4}+2}$ fails for $m\geqslant 6$, so we got a contradiction in this case.

In the remaining case $m=4$ we get from (18), (19), (20):

$$\sum_i \gamma_i^2 = 8+2\alpha\beta,\qquad \sum_i \gamma_i^2\theta_i = -2\alpha\beta-2\sqrt{6},\qquad \sum_i \gamma_i^2\theta_i^2 = 2\alpha\beta-4-(4\mp 2)\sqrt{6},$$

where $\alpha,\beta\in\mathbb{Z}_{\geqslant 0}$, $\alpha+\beta=2$. Comparing $\left|\sum_i \gamma_i^2\theta_i^2\right|$ with $\sum_i \gamma_i^2$ as above we receive that $\alpha=\beta=1$ and $\sum_i \gamma_i^2\theta_i^2 = -2-2\sqrt{6}=\sum_i \gamma_i^2\theta_i$, $\sum_i \gamma_i^2=10$.

It follows easily from [Lo, Theorem 2] that there is a unique way to represent $-2-2\sqrt{6}$ as a sum of 10 roots of unity, namely

$$-2-2\sqrt{6}=2(-1)+2\zeta_{24}^{7}+2\zeta_{24}^{-7}+2\zeta_{24}^{11}+2\zeta_{24}^{-11},$$

where $\zeta_{24}=e^{\pi i/12}$. Since this set of roots of unity is not closed under squaring, we have a contradiction. Theorem 4.1 is proved. □

Remark 4.3. One can give an alternative proof of Theorem 4.1 for $m \geqslant 6$ using only (18) and (19). For this one deduces from [Lo] that at least $2\alpha\beta + m\sqrt{\frac{1}{3}\left(\frac{m^2}{4}+2\right)}$ roots of unity are required in order to represent $-2\alpha\beta - \frac{m}{2}\sqrt{\frac{m^2}{4}+2}$ as their sum and obtains a contradiction with (18).

4.4. The ring $K(1, 0, 2, 0)$. The fusion categories $\mathcal{C}$ with $K(\mathcal{C}) = K(1, 0, 2, 0)$ were classified in [HH]. We sketch here an alternative argument for this classification. Recall that $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ has a natural structure of separable algebra, see e.g. [DMNO, Lemma 3.5]. Moreover, [DMNO, Theorem 4.10] implies that this algebra has a separable subalgebra I_0 with $\mathrm{FPdim}(I_0) = \frac{1}{2}\,\mathrm{FPdim}(I(\mathbf{1}))$. This forces $I_0 = \mathbf{1} \oplus A$. Thus $F(I_0) = 2\mathbf{1} \oplus X \in \mathcal{C}$ has a structure of separable algebra. This algebra has to be decomposable, so $\mathbf{1} \oplus X \in \mathcal{C}$ has a structure of separable algebra. It is easy to compute the "principal graph" of the category of $\mathbf{1} \oplus X$-modules in $\mathcal{C}$; it is precisely the Dynkin diagram of type E_6. Now the results of [HH] follow from the classification of subfactors of type E_6, at least in the case when the category $\mathcal{C}$ admits a unitary structure.

4.5. Proof of Theorem 1.1. Let $\mathcal{C}$ be a fusion category of rank 3 admitting a pivotal structure. Then by Theorems 3.3 and 4.1 the Grothendieck ring $K(\mathcal{C})$ is isomorphic to one of the following: $K(\mathbb{Z}/3\mathbb{Z})$, $K(1, 1, 1, 0)$, $K(1, 0, 0, 0)$, $K(1, 0, 1, 0)$ or $K(1, 0, 2, 0)$. For the first 4 rings the result follows from the discussion in [O3, Section 4], and for the last ring the result follows from [HH, Theorem 1]. □

5. Spherical Structures

5.1. Sphericalization. Let $\mathcal{C}$ be a fusion category. A *sphericalization* $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is defined in [ENO1, Remark 3.1]. Thus $\tilde{\mathcal{C}}$ is a spherical fusion category together with a tensor functor $\tilde{F} \colon \tilde{\mathcal{C}} \to \mathcal{C}$. The category $\mathcal{C}$ itself has a spherical structure if and only if the functor $\tilde{F}$ admits a tensor section $\mathcal{C} \to \tilde{\mathcal{C}}$. Equivalently, $\tilde{\mathcal{C}}$ should contain a fusion subcategory such that restriction of $\tilde{F}$ to this subcategory is an equivalence.

The functor $\tilde{F}$ has the following properties: it maps simple objects to simple objects and for any $X \in \mathcal{O}(\mathcal{C})$ there are precisely two objects $X_+, X_- \in \mathcal{O}(\tilde{\mathcal{C}})$ with $\tilde{F}(X_\pm) = X$ (the choice of X_+ and X_- is arbitrary except for $\mathbf{1}_+ = \mathbf{1}$). It follows that $\mathbf{1}_- \otimes \mathbf{1}_- = \mathbf{1}$, $\dim(\mathbf{1}_-) = -1$, and $X_\pm \otimes \mathbf{1}_- = \mathbf{1}_- \otimes X_\pm = X_\mp$. The following observation [T, proof of Lemma 2.3] is quite useful:

(a) If $X \in \mathcal{O}(\mathcal{C})$ is self-dual, then both X_+ and X_- are self-dual.

Let $\varepsilon = [\mathbf{1}_-] \in K(\tilde{\mathcal{C}})$; clearly ε is a central element. Observe that $K(\tilde{\mathcal{C}})/\langle \varepsilon = 1\rangle$ with a basis consisting of images of $[X_+]$ is isomorphic to $K(\mathcal{C})$ as a based ring (via the map $[X_+] \mapsto [X]$). Consider the ring $\tilde{K}(\mathcal{C}) = K(\tilde{\mathcal{C}})/\langle \varepsilon = -1\rangle$. This is a ring over $\mathbb{Z}$ with a basis consisting of images of $[X_+]$. In a sense this is just a basis up to signs since the signs of basis elements depend on the choice of $X_\pm$ above. The following statements are easy to verify:

(b) Let N be a structure constant of $K(\mathcal{C})$ and let $\tilde{N}$ be the corresponding structure constant of $\tilde{K}(\mathcal{C})$ (recall that the bases of both rings are labeled by the same set $\mathcal{O}(\mathcal{C})$). Then

$$|\tilde{N}| \leqslant N, \quad \text{and} \quad \tilde{N} \equiv N \pmod 2.$$

(c) The category $\mathcal{C}$ has a spherical structure if and only if there is a choice of signs of basis elements of $\tilde{K}(\mathcal{C})$ such that the corresponding structure constants equal.

5.2. Quadratic rings. Let us consider a based ring $K(k, l, m, n)$ and let $p(t)$ be the characteristic polynomial of operator R, see Section 3.2. We have the following possibilities for factorization of $p(t)$ over $\mathbb{Q}$:

(i) $p(t)$ is irreducible. We say that $K(k, l.m, n)$ is *cubic* in this case.

(ii) $p(t)$ has irreducible quadratic factor. In this case we say that $K(k, l.m, n)$ is *quadratic.*

(iii) $p(t)$ factorizes into three linear factors. We say that $K(k, l, m, n)$ is *rational.*

If $K(\mathcal{C}) = K(k, l, m, n)$ is rational, then $\mathrm{FPdim}(\mathcal{C})$ is integer, so $\mathcal{C}$ is spherical by [ENO1, Proposition 8.24]. We also have the following

Theorem 5.1. *Let $\mathcal{C}$ be a fusion category such that $K(\mathcal{C}) = K(k, l, m, n)$ is quadratic. Then $\mathcal{C}$ has a spherical structure.*

Proof. If $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$ then $\mathcal{C}$ has spherical structure by [ENO1, Proposition 8.24]. Thus we will assume for the rest of the proof that $\mathrm{FPdim}(\mathcal{C}) \notin \mathbb{Z}$. The polynomial $p(t)$ factorizes as $p(t) = (t-\gamma)(t^2 - \alpha t + \beta)$, where $\alpha, \beta, \gamma \in \mathbb{Z}$ and the second factor is irreducible. From the decomposition

$$(t-\gamma)(t^2 - \alpha t + \beta) = t^3 - (\alpha+\gamma)t^2 + (\beta + \alpha\gamma)t - \gamma\beta$$

we find

$$b = \alpha + \gamma, \quad c = \beta + \alpha\gamma = \gamma\beta, \quad \beta = \alpha\frac{\gamma}{\gamma - 1}. \tag{21}$$

In particular, we see that $\gamma - 1$ is a divisor of α and $\beta = \frac{\alpha}{\gamma-1}\gamma \geqslant \gamma$.

We now consider the sphericalization $\tilde{\mathcal{C}}$ and the ring $\tilde{K}(\mathcal{C})$. Using 5.1 (a) and (b) we see that $\tilde{K}(\mathcal{C}) = K(\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ (see (8)), where

$$|\tilde{k}| \leqslant k, \;\; |\tilde{l}| \leqslant l, \;\; |\tilde{m}| \leqslant m, \;\; |\tilde{n}| \leqslant n; \;\; k - \tilde{k},\, l - \tilde{l},\, m - \tilde{m},\, n - \tilde{n} \text{ are even.}$$

By changing the signs of basis elements of $\tilde{K}(\mathcal{C})$ we can (and will) assume that

$$\tilde{k} + \tilde{n} \geqslant 0 \quad \text{and} \quad \tilde{l} + \tilde{m} \geqslant 0.$$

We will generally use tilde for notations associated with $\tilde{K}(\mathcal{C})$ parallel to those in $K(\mathcal{C})$, for example $\tilde{R}$, $\tilde{p}(t)$ etc. Notice that the sum of inverse roots of $\tilde{p}(t)$ equals 1: indeed the formal codegrees of $K(\tilde{\mathcal{C}})$ are precisely doubled formal codegrees of $K(\mathcal{C})$ and doubled roots of $\tilde{p}(t)$, so the result follows from Proposition 2.10. Thus

$$\tilde{p}(t) = t^3 - \tilde{b}t^2 + \tilde{c}t - \tilde{c}.$$

It follows from (10) that $\tilde{b} \leqslant b$. Moreover, the equality $\tilde{b} = b$ implies $\tilde{k} = k$, $\tilde{l} = l$, $\tilde{m} = m$, $\tilde{n} = n$, so the category $\mathcal{C}$ is spherical by 5.1 (c). Thus in the rest of this proof we will assume that $\tilde{b} < b$ and derive a contradiction.

Observe that the dimension homomorphism $K(\tilde{\mathcal{C}}) \to k$ factors through $\tilde{K}(\mathcal{C})$, so the degree of the dimension field $L(\tilde{\mathcal{C}})$ over $\mathbb{Q}$ is at most three. On the other hand it follows from Corollary 2.15 that $L(\tilde{\mathcal{C}})$ contains the splitting field of $t^2-\alpha t+\beta$; it follows that $L(\tilde{\mathcal{C}})$ is precisely this field. Thus we have two homomorphisms $\tilde{K}(\mathcal{C}) \to k$ with values in $L(\tilde{\mathcal{C}})$: the dimension homomorphism and its Galois conjugate. Thus the remaining homomorphism must be Galois invariant, so it lands in $\mathbb{Z}$. In particular, the polynomial $\tilde{p}(t)$ must be reducible.

Assume that $\tilde{p}(t)$ has three integer roots $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$. Two of these roots corresponding to the dimension homomorphism and its Galois conjugate must be equal, say $\tilde{f}_1 = \tilde{f}_2$. From the equation $\frac{1}{\tilde{f}_1}+\frac{1}{\tilde{f}_2}+\frac{1}{\tilde{f}_3} = 1$ we find that $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = (3, 3, 3)$ or $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = (4, 4, 2)$. Hence $\tilde{b} = 9$ or 10. From $\tilde{b} = 9$ we get a contradiction using (10) and (7). Thus $\tilde{b} = 10$ and $\dim(\mathcal{C}) = \tilde{f}_1 = 4$. Therefore γ divides 4 by Corollary 2.14. Observe that $\gamma = f_E$, where E is the one dimensional representation of $K(\mathcal{C})$ corresponding to the $\mathbb{Z}$-valued character χ. Thus $\gamma = 1 + s^2 + t^2$, where $s = \chi(X) \in \mathbb{Z}$ and $t = \chi(Y) \in \mathbb{Z}$, see [O4, Example 2.4]. It follows that $\gamma = 2$ and in view of Remark 3.2 (i) we can assume that $s = 0$. Then (8) implies that $l = 0$ and $K(\mathcal{C}) = K(1, 0, m, 0)$. However in this case $\mathcal{C}$ is near-group category, so it is spherical by [T, Theorem 1.3]. Hence Theorem 5.1 holds in this case.

Thus we will assume that $\tilde{p}(t)$ is a product of linear factor and of irreducible quadratic factor

$$\tilde{p}(t) = (t-\tilde{\gamma})(t^2-\tilde{\alpha}t+\tilde{\beta}).$$

Note that $\dim(\mathcal{C})$ is one of the roots of $t^2-\tilde{\alpha}t+\tilde{\beta}$, and the splitting fields of $p(t)$ and of $\tilde{p}(t)$ coincide

Clearly,

$$\tilde{b} = \tilde{\alpha}+\tilde{\gamma}, \quad \tilde{c} = \tilde{\beta}+\tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\beta}, \quad \tilde{\beta} = \tilde{\alpha}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}. \tag{22}$$

We have $\tilde{\alpha}+\tilde{\gamma} = \tilde{b} < b = \alpha+\gamma$ and $\frac{\tilde{\gamma}}{\tilde{\gamma}-1} \leqslant 2$ whence

$$\tilde{\beta} = \tilde{\alpha}\frac{\tilde{\gamma}}{\tilde{\gamma}-1} < 2(\alpha+\gamma-\tilde{\gamma}) = 2\left(\beta\frac{\gamma-1}{\gamma}+\gamma-\tilde{\gamma}\right) < 4\beta$$

(recall that $\beta \geqslant \gamma$). Since $\frac{\tilde{\beta}}{\beta} \in \mathbb{Z}$ by Corollary 2.14, we have the following possibilities:

Case 1: $\tilde{\beta} = 3\beta$. In this case $2\tilde{\alpha} \geqslant \tilde{\alpha}\frac{\tilde{\gamma}}{\tilde{\gamma}-1} = 3\alpha\frac{\gamma}{\gamma-1}$ whence $\tilde{\alpha} \geqslant \frac{3}{2}\alpha+\frac{3}{2}\frac{\alpha}{\gamma-1}$. Notice that by Corollary 2.14 γ divides $\dim(\mathcal{C})$, hence γ^2 divides the norm $\tilde{\beta}$ of $\dim(\mathcal{C})$, which is equivalent to $\frac{3\alpha}{\gamma(\gamma-1)} \in \mathbb{Z}$. If $\frac{3\alpha}{\gamma(\gamma-1)} \geqslant 2$ then $\frac{3}{2}\frac{\alpha}{\gamma-1} \geqslant \gamma$, so $\tilde{\alpha} > \alpha+\gamma$, which contradicts to the inequality $\tilde{\alpha}+\tilde{\gamma} < \alpha+\gamma$. If $\frac{3\alpha}{\gamma(\gamma-1)} = 1$ then $\alpha = \frac{1}{3}\gamma(\gamma-1)$, $\beta = \frac{1}{3}\gamma^2$. Thus γ is divisible by 3, so $\frac{\alpha^2}{\beta} = \frac{1}{3}(\gamma-1)^2 \notin \mathbb{Z}$, whence $\mathrm{FPdim}(\mathcal{C})$ is not a d-number and we have a contradiction with Proposition 2.2.

Case 2: $\tilde{\beta} = 2\beta$. In this case $2\tilde{\alpha} \geqslant \tilde{\alpha}\frac{\tilde{\gamma}}{\tilde{\gamma}-1} = 2\alpha\frac{\gamma}{\gamma-1}$ whence $\tilde{\alpha} \geqslant \alpha+\frac{\alpha}{\gamma-1}$. Notice that by Corollary 2.14 γ divides $\dim(\mathcal{C})$, hence γ^2 divides the norm $\tilde{\beta}$ of $\dim(\mathcal{C})$, which is equivalent to $\frac{2\alpha}{\gamma(\gamma-1)} \in \mathbb{Z}$. If $\frac{2\alpha}{\gamma(\gamma-1)} \geqslant 2$ then $2\frac{\alpha}{\gamma-1} \geqslant \gamma$, so $\tilde{\alpha} > \alpha+\gamma$, which contradicts to the inequality $\tilde{\alpha}+\tilde{\gamma} < \alpha+\gamma$. If $\frac{2\alpha}{\gamma(\gamma-1)} = 1$ then $\alpha = \frac{1}{2}\gamma(\gamma-1)$, $\beta = \frac{1}{2}\gamma^2$. Thus γ is even, so $\frac{\alpha^2}{\beta} = \frac{1}{2}(\gamma-1)^2 \notin \mathbb{Z}$, whence $\mathrm{FPdim}(\mathcal{C})$ is not a d-number and we have a contradiction with Proposition 2.2.

Case 3: $\tilde\beta = \beta$. First of all we claim that $\tilde\gamma < \gamma$. Indeed inequality $\tilde\gamma \geqslant \gamma$ implies $\tilde\alpha = \alpha \dfrac{1+\frac{1}{\gamma-1}}{1+\frac{1}{\tilde\gamma-1}} \geqslant \alpha$, so $\tilde\alpha + \tilde\gamma \geqslant \alpha + \gamma$.

The splitting fields of the polynomials $t^2 - \alpha t + \beta$ and $t^2 - \tilde\alpha t + \tilde\beta$ should coincide, equivalently the ratio of their discriminants $\dfrac{\alpha^2 - 4\beta}{\tilde\alpha^2 - 4\tilde\beta} = \dfrac{\frac{\alpha^2}{\beta} - 4}{\frac{\tilde\alpha^2}{\beta} - 4}$ should be a square of rational number (notice that the numerator and denominator of the last expression are integers since $\dim(\mathcal{C})$ and $\mathrm{FPdim}(\mathcal{C})$ are d-numbers). Thus there exist a square free integer d and positive integers x, y such that

$$\frac{\alpha^2}{\beta} - 4 = \left(\frac{\gamma-1}{\gamma}\right)^2 \beta - 4 = dx^2, \quad \frac{\tilde\alpha^2}{\beta} - 4 = \left(\frac{\tilde\gamma-1}{\tilde\gamma}\right)^2 \beta - 4 = dy^2. \tag{23}$$

Henceforth we have

$$\frac{dx^2+4}{dy^2+4} = \left(\frac{1+\frac{1}{\tilde\gamma-1}}{1+\frac{1}{\gamma-1}}\right)^2.$$

Let $\dfrac{1+\frac{1}{\tilde\gamma-1}}{1+\frac{1}{\gamma-1}} = \frac{A}{B}$, where $\frac{A}{B}$ is in its lowest terms. Notice that inequalities $2 \leqslant \tilde\gamma < \gamma$ imply $1 < \frac{A}{B} < 2$. The equality $\frac{dx^2+4}{dy^2+4} = \left(\frac{A}{B}\right)^2$ is equivalent to

$$d(Bx - Ay)(Bx + Ay) = 4(A^2 - B^2).$$

Since d is square free, the numbers $Bx - Ay$ and $Bx + Ay$ are both even. We set $q = \frac{Bx-Ay}{2}$, so $\frac{Bx+Ay}{2} = \frac{A^2-B^2}{dq}$. Hence

$$Bx = \frac{A^2-B^2}{dq} + q = \frac{A^2 - B^2 + dq^2}{dq}, \quad Ay = \frac{A^2-B^2}{dq} - q = \frac{A^2 - B^2 - dq^2}{dq}. \tag{24}$$

These equalities imply that $A^2 + B^2 + dq^2$ is divisible by both A and B, so it is divisible by AB. We claim that

$$2 < \frac{A^2 + B^2 + dq^2}{AB} < 4.$$

Indeed, from (24) we get $A^2 - B^2 - dq^2 = Aydq > 0$, so $\frac{A}{B} - \frac{B}{A} > \frac{dq^2}{AB}$, whence $\frac{dq^2}{AB} < \frac{3}{2}$ (since $\frac{3}{2}$ is the maximal value of function $x - \frac{1}{x}$ for $x \in [1, 2]$). Now $\frac{A^2+B^2+dq^2}{AB} = \frac{A}{B} + \frac{B}{A} + \frac{dq^2}{AB}$, where $\frac{A}{B} + \frac{B}{A} \in (2, \frac{5}{2})$ (since $\frac{A}{B} \in (1, 2)$) and the desired inequality is established. The previous remarks imply

$$\frac{A^2 + B^2 + dq^2}{AB} = 3$$

and, eliminating dq^2 we get

$$x = \frac{3A - 2B}{dq}, \quad y = \frac{2A - 3B}{dq}. \tag{25}$$

Thus dq is divisor of $5A = 3(3A - 2B) - 2(2A - 3B)$ and $5B = 2(3A - 2B) - 3(2A - 3B)$; since A and B are coprime we see that dq divides 5. It follows that $dq^2 = 1$, 5 or 25.

Subcase $dq^2 = 25$. The equality $A^2+B^2+25 = 3AB$ is equivalent to $(2A-3B)^2-5B^2 + 100 = 0$, which implies that both A and B are divisible by 5. This is a contradiction.

Subcase $dq^2 = 1$. The equality $A^2 + B^2 + 1 = 3AB$ is equivalent to $\frac{1}{B^2} = 3\frac{A}{B} - \frac{A^2}{B^2} - 1$, which implies that $\frac{1}{B^2} > 1$ (since the minimal value of the function $3x - x^2 - 1$ on the interval $[1, 2]$ is 1), which is a contradiction.

Subcase $dq^2 = 5$. As in the previous case we get $\frac{5}{B^2} > 1$ whence $\frac{A}{B} = \frac{3}{2}$. Then (25) gives $y = 0$ and we get a final contradiction. □

5.3. Examples for cubic rings. It is easy to see that the ring $K(k, l, m, n)$ is cubic if and only if $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field. The following result is useful in order to determine whether cubic $K(k, l, m, n)$ passes the cyclotomic test:

Proposition 5.2. *Assume that $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field. This field is abelian if and only if c is a square of integer.*

Proof. Observe that the matrix of the trace form on the ring $K(k, l, m, n)$ with respect to its basis coincides with the matrix (9). Thus the discriminant of the field $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is c modulo rational squares. Finally it is well known that a cubic field is abelian if and only if its discriminant is square. □

Remark 5.3. The proof of Proposition 5.2 shows that the primes ramified in the field $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ are divisors of c. Of course it is possible that some prime divisor of c does not ramify. However if $K(k, l, m, n)$ passes the d-number test (see Proposition 2.2) then a root of $p(t) = t^3 - bt^2 + ct - c$ is $\sqrt[3]{c}$ modulo units, so if $c = \prod_i p_i^{\alpha_i}$ (with distinct primes p_i) and α_i is not divisible by 3 then p_i is ramified in $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The cubic based rings $K(k, l, m, n)$ that pass both the cyclotomic and d-number tests from Section 2.1 are very sparse (but probably there are infinitely many of them). A complete list of such rings with $l \leqslant k \leqslant 100000$ obtained as a result of computer search includes categorifiable ring $K(1, 1, 1, 0)$ and the following rings:

k, l, m, n	k, l, m, n	k, l, m, n
29, 13, 62, 7	1493, 863, 2529, 530	34487, 23653, 50489, 16090
83, 77, 91, 70	2339, 323, 16906, 49	43037, 25271, 74851, 13924
305, 179, 530, 99	7579, 4063, 14136, 2179	47603, 29251, 77448, 17987
409, 331, 137, 566	8401, 5099, 13874, 3075	54559, 41609, 71568, 31711
1133, 169, 7624, 21	8621, 473, 157182, 23	86593, 16571, 453173, 3042
1373, 31, 60753, 2	20341, 3887, 106288, 773	95705, 14221, 641435, 2506

Remark 5.4. The following observations (especially the second one) are useful in the above mentioned computer search:

(i) If b^3 is divisible by c then $p(t)$ is irreducible. Indeed, assume that $p(t)$ is reducible, $p(t) = (t - \gamma)(t^2 - \alpha t + \beta)$. Then (21) implies that $s := \frac{\alpha}{\gamma - 1}$ is an integer and we have $b = \alpha + \gamma = s\gamma + \gamma - s$, $c = \gamma\beta = \gamma^2 s$. Hence $\frac{b^3}{c} \in \mathbb{Z}$ is equivalent to $3\frac{s^2}{\gamma} + \frac{\gamma}{s} + 3\frac{s}{\gamma} - \frac{s^2}{\gamma^2} \in \mathbb{Z}$. This condition implies that γ and s must have the same p-adic valuation for each prime p, whence $\gamma = s$. Thus, $b = s^2$, which is impossible since by (10) b is congruent to 2 or 3 modulo 4.

(ii) k and l should be odd (which implies that c is odd and $b \equiv 2 \pmod 4$). Indeed if k or l is even then (7) and (12) imply that c is even, hence b is even by the d-number test. Then (10) shows that $b \equiv 2 \pmod 4$. Using Proposition 5.2 and the d-number test we see that c is divisible by 4 but not by 8. Hence by Remark 5.3 the field $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is ramified at 2. This is a contradiction since the only extension of 2-adic numbers $\mathbb{Q}_2$ with Galois group $\mathbb{Z}/3\mathbb{Z}$ is unramified.

We describe now the computer computations which show that a categorification of the rings above is spherical (and hence does not exist).

1) First of all we can give an estimate of $\tilde{c}$. Using (12) and (7) we see that

$$\tilde{c} = 4\tilde{b} - 9 + (\tilde{k}\tilde{l} - \tilde{m}\tilde{n})^2 + (2(\tilde{k}^2 - \tilde{l}\tilde{m}) - 1)^2 = 4\tilde{b} - 9 + (\tilde{k}\tilde{l} - \tilde{m}\tilde{n})^2 + (2(\tilde{l}^2 - \tilde{k}\tilde{n}) - 1)^2.$$

Hence,

$$\tilde{c} \leqslant \min(4b - 9 + (kl + mn)^2 + (2k^2 + 2lm + 1)^2,\ 4b - 9 + (kl + mn)^2 + (2l^2 + 2kn + 1)^2).$$

Example 5.5. For the ring $K(54559, 41609, 71568, 31711)$ we obtain

$$\frac{\tilde{c}}{c} \leqslant 11132295.1 < 15^6,$$

and for the ring $K(95705, 14221, 641435, 2506)$ we obtain

$$\frac{\tilde{c}}{c} < 144.6 < 13^2 < 3^6.$$

2) Recall that by Corollary 2.14 $\tilde{c}$ should be divisible by c; also the fields $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[t]/(p(t))$ and $K(\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[t]/(\tilde{p}(t))$ should be isomorphic (since by Corollary 2.15 $K(k, l, m, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ should be contained in the dimension field $L(\tilde{\mathcal{C}})$, which is contained in $K(\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}) \otimes_{\mathbb{Z}} \mathbb{Q}$). In view of Remark 5.3 this shows that the prime factors of $\tilde{c}$ which do not appear in the prime decomposition of c should have exponents divisible by 6. Using the estimate for $\tilde{c}$ above we can get a list of possible values of $\tilde{c}$ (note that in view of Remark 5.4 (ii), $\tilde{c}$ should be odd).

Example 5.6. For $K(54559, 41609, 71568, 31711)$ we have $c = 7^2 \cdot 127^2 \cdot 2791^2$. Thus the possible values of $\tilde{c}/c$ are: $1, 3^6, 5^6, 7^6, 9^6, 11^6, 13^6, 7^2, 7^2 \cdot 3^6, 7^2 \cdot 5^6, 7^4, 7^4 \cdot 3^6, 7^8, 7^2 \cdot 127^2, 127^2, 2791^2$.

For the ring $K(95705, 14221, 641435, 2506)$ we have $c = 3^6 \cdot 13^2 \cdot 151^2 \cdot 4861^2$ and the possible values of $\tilde{c}/c$ are $1, 3^2, 3^4$.

3) For each value of $\tilde{c}$ the possible values of $\tilde{b}$ satisfy the following: $\tilde{b}^3$ is divisible by $\tilde{c}$ (and hence by c), $0 < \tilde{b} \leqslant b$, and $\tilde{b} \equiv b \pmod 4$; this can be enumerated efficiently by computer.

Example 5.7. For the ring $K(54559, 41609, 71568, 31711)$ we get

$$\tilde{b} = b - i \cdot 4 \cdot 7 \cdot 127 \cdot 2791, \quad i \in \mathbb{Z},\ \ 0 \leqslant i \leqslant 2040,$$

and for the ring $K(95705, 14221, 641435, 2506)$ we get

$$\tilde{b} = b - i \cdot 4 \cdot 3^2 \cdot 13 \cdot 151 \cdot 4861, \quad i \in \mathbb{Z}, \ \ 0 \leqslant i \leqslant 1279.$$

4) Then for each $\tilde{c}$ and $\tilde{b}$ we can compute the discriminant of $\tilde{p}(t)$ and discard all the cases where the field $\mathbb{Q}[t]/(\tilde{p}(t))$ is not abelian. We found that the only possibilities that are not discarded are either $\tilde{b} = b$, $\tilde{c} = c$ (which implies pseudo-unitarity of a categorification of $K(k, l, m, n)$) or the following two cases: $K(29, 13, 62, 7)$ with $\tilde{b} = 378$, $\tilde{c} = c = 35721$, and $K(83, 77, 91, 70)$ with $\tilde{b} = 1026$, $\tilde{c} = c = 263169$.

5) Finally we deal with the two exceptional cases above.

(a) In the case of $K(29, 13, 62, 7)$ there exists no ring $K(\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ consistent with $\tilde{b} = 378$ and $\tilde{c} = 35721$ (we can find $\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}$ from equations $(\tilde{k} + \tilde{n})^2 + (\tilde{l} + \tilde{m})^2 = \tilde{b} - 9$ and $(3\tilde{k} - \tilde{n})^2 + (3\tilde{l} - \tilde{m})^2 = \tilde{b} - 1$, which follow from (10) and (7); up to signs the only solution is $K(7, 1, 14, 5)$, which is incompatible with the value of $\tilde{c}$). Note that in this case the fields $\mathbb{Q}[t]/(p(t))$ and $\mathbb{Q}[t]/(\tilde{p}(t))$ are isomorphic (one isomorphism sends $t \in \mathbb{Q}[t]/(p(t))$ to $-\frac{13}{9}t^2 + \frac{881}{3}t - 294 \in \mathbb{Q}[t]/(\tilde{p}(t))$).

(b) In the case of $K(83, 77, 91, 70)$ the fields $\mathbb{Q}[t]/(p(t))$ and $\mathbb{Q}[t]/(\tilde{p}(t))$ are not isomorphic (here is an easy way to verify this: let t_1, t_2, t_3 be roots of $p(t)$ and let $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3$ be the roots of $\tilde{p}(t)$. If the fields above are isomorphic then one of the numbers $t_1\tilde{t}_1 + t_2\tilde{t}_2 + t_3\tilde{t}_3$ or $t_1\tilde{t}_1 + t_2\tilde{t}_3 + t_3\tilde{t}_2$ would be integer. However a numerical computation shows that this is not the case.). Note that in this case the ring $K(1, 1, 23, -22)$ is compatible with the values of $\tilde{b}$ and $\tilde{c}$.

We summarize the results of this section as follows:

Theorem 5.8. *Assume that a based ring $K(k, l, m, n)$ with $k \geqslant l$ admits a non-spherical categorification. Then $k > 100{,}000$.*

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