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ON MONODROMY IN FAMILIES OF ELLIPTIC CURVES OVER $\mathbb{C}$

SERGE LVOVSKI

ABSTRACT. We show that if we are given a smooth non-isotrivial family of curves of genus 1 over $\mathbb{C}$ with a smooth base B for which the general fiber of the mapping $J\colon B \to \mathbb{A}^1$ (assigning j-invariant of the fiber to a point) is connected, then the monodromy group of the family (acting on $H^1(\cdot, \mathbb{Z})$ of the fibers) coincides with $\mathrm{SL}(2, \mathbb{Z})$; if the general fiber has $m \geqslant 2$ connected components, then the monodromy group has index at most $2m$ in $\mathrm{SL}(2, \mathbb{Z})$. By contrast, in *any* family of hyperelliptic curves of genus $g \geqslant 3$, the monodromy group is strictly less than $\mathrm{Sp}(2g, \mathbb{Z})$.

Some applications are given, including that to monodromy of hyperplane sections of Del Pezzo surfaces.

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KEY WORDS AND PHRASES. Monodromy, elliptic curve, hyperelliptic curve, j-invariant, braid, Del Pezzo surface.

INTRODUCTION

It is believed that if fibers in a family of algebraic varieties "vary enough" then the monodromy group acting on the cohomology of the fiber should be in some sense big. Quite a few results have been obtained in this direction. See for example [4] for families of elliptic curves, [10] for families of hyperelliptic curves, [2] for families of abelian varieties (see also [11] for abelian varieties in the arithmetic situation). In the cited papers cohomology means "étale cohomology with finite coefficients". In this paper we address the question of "big monodromy" for families of elliptic curves over $\mathbb{C}$ and singular cohomology.

The main result of the paper (Proposition 4.2) asserts that *if* $\pi\colon \mathcal{X} \to B$ *is a smooth non-isotrivial family of curves of genus* 1 *over* $\mathbb{C}$ *and if the general fiber of its "J-map"* $J_{\mathcal{X}}\colon B \to \mathbb{A}^1$ *(assigning to each point of the base the j-invariant of the fiber) is connected, then the monodromy group of the family* $\mathcal{X}$ *is the entire group* $\mathrm{SL}(2, \mathbb{Z})$*, and if the general fiber has* $m \geqslant 2$ *connected components, then the monodromy group of the family* $\mathcal{X}$ *is a subgroup of index at most* $2m$ *in* $\mathrm{SL}(2, \mathbb{Z})$. Here, by monodromy group we mean the image of the natural mapping $\pi_1(B, b) \to \mathrm{SL}(H^1(\mathcal{X}_b, \mathbb{Z}))$, where $b \in B$ is a general enough point and $\mathcal{X}_b$ is the fiber.

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An immediate consequence of this proposition is that *in any non-isotrivial family of curves of genus* 1*, the monodromy group has finite index in* $\mathrm{SL}(2, \mathbb{Z})$ (Corollary 4.4). This requires some comments.

The above assertion is similar to a well-known result about elliptic curves over number fields, that is, to Serre's Theorem 3.2 from Chapter IV of [14]. It is possible that one can prove our Corollary 4.4 by imitating, *mutatis mutandis*, Serre's proof of this theorem or even derive it from Serre's theorem or similar arithmetical results. One merit of the approach presented in this paper is that the proofs are very simple and elementary. One should add that the similarity between arithmetic and geometric situations is not absolute. For example, Theorem 5.1 from [10] could suggest that, over $\mathbb{C}$, the monodromy group for some families of hyperelliptic curves of genus g should be the entire $\mathrm{Sp}(2g, \mathbb{Z})$. However, as we show in Proposition 5.2, for *any* family of hyperelliptic curves of genus $g \geqslant 3$ over $\mathbb{C}$ the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of the fiber is a proper subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$.

Our main result has three simple consequences, which are presented in Section 4. First, any smooth family of curves of genus 1 over a smooth base with commutative fundamental group, must be isotrivial (Proposition 4.5). Second, for non-isotrivial families we obtain an upper bound on the index of the monodromy group in $\mathrm{SL}(2, \mathbb{Z})$ in terms of the number of generators of π_1 of the base (Proposition 4.6). Third, in the case of smooth elliptic surfaces we use Miranda's results from [13] to obtain an upper bound on the index of monodromy group in terms of singular fibers (see Proposition 4.10). It should be noted that if all the singular fibers are of the type I_1, then the monodromy group is the entire $\mathrm{SL}(2, \mathbb{Z})$, see the book [7] by R. Friedman and J. W. Morgan (Chapter II, Theorem 3.8).

In Section 5 we prove the above mentioned result about families of hyperelliptic curves of genus 3 or higher.

In Section 6, we derive from our main result that the hyperplane monodromy group of a smooth Del Pezzo surface (or, for Del Pezzos of degree 2, the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of smooth elements of the anticanonical linear system) is the entire $\mathrm{SL}(2, \mathbb{Z})$ (Proposition 6.1). Observe that, in view of Proposition 5.2, Proposition 6.1 cannot be extended to surfaces with hyperelliptic hyperplane sections.

Sections 1 through 3 are devoted to auxiliary material.

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Notation and conventions. All algebraic varieties are defined over $\mathbb{C}$, the only exception being the discussion of quadratic twists in Section 3. If X is an algebraic variety, then X_{sm} and X_{sing} are its smooth and singular loci.

When we say "a general X has property Y", this always means "property Y holds for a Zariski open and dense set of X's". The word "generic" is used in the scheme-theoretic sense.

If B is an algebraic variety and $\pi\colon \mathcal{X} \to B$ is a proper and flat morphism such that a general fiber of π is a smooth curve of genus 1, we will say that π is a *family of curves of genus* 1. If, in addition, the morphism f is smooth, we will say that

π (or just $\mathcal{X}$ if there is no danger of confusion) is a *smooth family*, or *a family of smooth varieties.* If $\pi\colon \mathcal{X} \to B$ is a family over B and $f\colon B' \to B$ is a morphism, then by $p'\colon \mathcal{X}_{B'} \to B'$ we mean the pullback of $\mathcal{X}$ along f.

By π_1 of an algebraic variety over $\mathbb{C}$ we always mean the fundamental group in the classical (complex) topology.

As usual, we put $\Gamma(2) = \{A \in \mathrm{SL}(2, \mathbb{Z})\colon A \equiv I \pmod 2\}$, where I is the identity matrix.

Following Miranda [13], we distinguish between curves of genus 1 and elliptic curves: by elliptic curve over a field K we mean a smooth projective curve over K of genus 1 with a distinguished K-rational point.

Similarly, by a *smooth family of curves of genus* 1 we will mean a smooth family $\pi\colon \mathcal{X} \to B$ such that its fibers are curves of genus 1, and by a *smooth family of elliptic curves* we mean a pair $(\mathcal{X}, s)$, where $\mathcal{X} \to B$ is a smooth family of curves of genus 1 and $s\colon B \to \mathcal{X}$ is a section.

To each curve C of genus 1 over a filed K, $\operatorname{char} K = 0$, one can assign its *$j$-invariant* $j(C) \in K$; recall that if C is (the smooth projective model of) the curve defined by the Weierstrass equation $y^2 = x^3 + px + q$, then

$$j(C) = 1728 \cdot \frac{4p^3}{4p^3 + 27q^2}. \tag{1}$$

Two curves of genus 1 over $\mathbb{C}$ are isomorphic if and only if their j-invariants are equal.

We say that a family over B is isotrivial if it becomes trivial after a pullback along a generically finite morphism $B_1 \to B$. For families of curves of genus 1 this is equivalent to the condition that j-invariants of all fibers are the same.

1. Generalities on Monodromy Groups

Suppose that B is an irreducible variety and $\pi\colon \mathcal{X} \to B$ is a family of smooth varieties.

If $b \in B_{\mathrm{sm}}$, $k \in \mathbb{N}$, and G is an abelian group, then the fundamental group $\pi_1(B_{\mathrm{sm}}, b)$ acts on $H^k(p^{-1}(b), G)$.

Definition 1.1. The image of $\pi_1(B_{\mathrm{sm}}, b)$ in $\mathrm{Aut}(H^k(p^{-1}(b), G))$ (corresponding to this action) will be called *monodromy group* of the family $\mathcal{X}$ at b and denoted $\mathrm{Mon}(\mathcal{X}, b)$ (we suppress the mention of k and G; there will be no danger of confusion).

Since B is irreducible, B_{sm} is path connected. Hence, if we fix once and for all the group $A = \mathrm{Aut}(H^k(p^{-1}(b_0), G))$ for some $b_0 \in B_{\mathrm{sm}}$, then all the groups $\mathrm{Mon}(\mathcal{X}, b)$ are conjugate in A; any such subgroup will be denoted by $\mathrm{Mon}(\mathcal{X})$.

In the sequel we will be working with families of smooth curves of genus g (in most cases g will be equal to 1) as fibers and monodromy action on H^1 of the fiber. Since monodromy preserves the intersection form, the subgroups $\mathrm{Mon}(\mathcal{X})$, where $\mathcal{X}$ is such a family, will be defined up to an inner automorphism of the group $\mathrm{Sp}(2g, \mathbb{Z})$ ($\mathrm{SL}(2, \mathbb{Z})$ if $g = 1$).

If $\pi\colon \mathcal{X} \to B$ is a non-smooth family, then by $\mathrm{Mon}(\mathcal{X})$ we mean $\mathrm{Mon}(\mathcal{X}|_U)$, where $U \subset B$ is the Zariski open subset over which π is smooth.

Proposition 1.2. *Suppose that B is an irreducible variety, $U \subset B$ is a non-empty Zariski open subset, and $\mathcal{X}$ is a smooth family over B. Then* $\operatorname{Mon}(\mathcal{X}|_U) = \operatorname{Mon}(\mathcal{X})$.

Proof. The result follows from the fact that, for any $b \in U \cap B_{\mathrm{sm}}$, the natural homomorphism $\pi_1(U \cap B_{\mathrm{sm}}, b) \to \pi_1(B_{\mathrm{sm}}, b)$ is epimorphic (see for example [8, 0.7(B) ff.]). □

Proposition 1.3. *Suppose that B' and B are smooth irreducible varieties and $\mathcal{X}$ is a smooth family over B. If $f \colon B' \to B$ is a dominant morphism such that a general fiber of f has m connected components, then $\operatorname{Mon}(\mathcal{X}_{B'})$ is conjugate to a subgroup of $\operatorname{Mon}(\mathcal{X})$, of index at most m. In particular, if $f \colon B' \to B$ is a dominant morphism such that a general fiber of f is connected, then $\operatorname{Mon}(\mathcal{X}_{B'})$ is conjugate to $\operatorname{Mon}(\mathcal{X})$.*

Proof. Immediate from the fact that f is a locally trivial bundle in the complex topology over a Zariski open subset of B (see [16, Corollary 5.1]). □

2. Some Remarks on 3-Braids

In this section, all topological terms will refer to the classical (complex) topology.

We begin with some remarks on 3-braids (see for example [9]).

If $\mathbb{C}^{(3)}$ is the configuration space of unordered triples of distinct points in the complex plane, then $\pi_1(\mathbb{C}^{(3)}) \cong B_3$ (the braid group with 3 strands). If $(u, v, w) \in \mathbb{C}^{(3)}$, we will write $B_3(u, v, w)$ instead of $\pi_1(\mathbb{C}^{(3)}, (u, v, w))$.

For any triple $(u, v, w) \in \mathbb{C}^{(3)}$, we denote by $X_{u,v,w}$ the elliptic curve which is the smooth projective model of the curve with the equation $y^2 = (x-u)(x-v)(x-w)$.

Any braid $\gamma \in B_3(u, v, w)$ can be represented by a homeomorphism $\varphi_\gamma \colon \mathbb{P}^1 \to \mathbb{P}^1$ (where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \supset \mathbb{C}$) such that $\varphi_\gamma(\{u, v, w\}) = \{u, v, w\}$ and $\varphi_\gamma(\infty) = \infty$. If $\pi \colon X_{u,v,w} \to \mathbb{P}^1 \supset \mathbb{C}$ is induced by the projection $(x, y) \mapsto x$, then there exists a unique homeomorphism $\tilde\varphi_\gamma \colon X_{u,v,w} \to X_{u,v,w}$ such that $\pi \circ \tilde\varphi_\gamma = \varphi_\gamma \circ \pi$ and φ_γ fixes $\pi^{-1}(\infty)$. The automorphism $\tilde\varphi_\gamma^* \colon H^1(X_{u,v,w}, \mathbb{Z}) \to H^1(X_{u,v,w}, \mathbb{Z})$ does not depend on the choice of the φ_γ representing γ; we put $\mu(\gamma) = \tilde\varphi_\gamma^*$.

Proposition 2.1. *If $\Gamma \in B_3(u, v, w)$ is the braid represented by the loop in $\mathbb{C}^{(3)}$ defined by the formula $t \mapsto (ue^{2\pi i t}, ve^{2\pi i t}, ve^{2\pi i t})$, $t \in [0; 1]$, then* $\mu(\Gamma) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Proof. Choose the points $u, v, w \in \mathbb{C}$ as in Figure 1. Now let A and B be the braids corresponding to the following closed paths in $\mathbb{C}^{(3)}$: in the path defining A, the point w stays where it is while u and v are swapped, u and v moving along small arcs close to the segment $[p, q]$ so that the composition of paths traveled by u and v defines a positively oriented simple closed curve. The braid B is defined similarly, with the point u staying put and the points v and w being exchanged; see Figure 1. The group $B_3(u, v, w)$ is generated by A and B, with the relation $ABA = BAB$.

For a basis in $H_1(X_{u,v,w}, \mathbb{Z})$ we choose the 1-cycles α and β that are obtained by lifting the closed paths α and β on Fig. 2 from $\mathbb{C}$ to $X_{u,v,w}$. Observe that the lifts of α and β are closed cycles indeed since analytic continuation of a germ of the function $\sqrt{(x-u)(x-v)(x-w)}$ along the path α results in the same germ, and similarly for β.

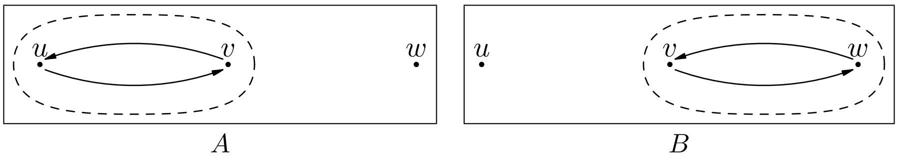

FIGURE 1. Two generators of $B_3(u,\, v,\, w)$. One may assume that homeomorphisms of $\mathbb{C}$ representing these braids are identity outside the dashed ovals.

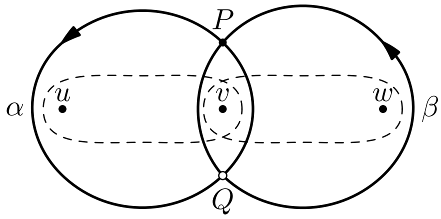

FIGURE 2. Projections of the cycles α and β to $\mathbb{C}$. Point P corresponds to their intersection point on $X_{u,v,w}$, point Q is just an apparent intersection point.

Denoting the action of A and B on $H_1(X_{u,v,w},\, \mathbb{Z})$ by the same letters A and B, it is clear that $A(\alpha) = \alpha$ and $B(\beta) = \beta$. Since A and B preserve the intersection pairing on $H_1(X_{u,v,w},\, \mathbb{Z})$, it is clear that, in the basis $(\alpha,\, \beta)$, the action of A and B on H_1 is given by matrices of the form

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \quad a,\, b \in \mathbb{Z}.$$

The relation $ABA = BAB$ implies that either $a = b = 0$ or $ab = -1$. The first case is impossible since the action of B_3 is non-trivial (see for example [1]). So, one of the integers a and b is equal to 1 and the other is equal to -1. Since $\Gamma = (AB)^3$, the result follows. □

3. Quadratic Twists and Monodromy

In this and the following section we will be studying monodromy groups acting on $H^1(\cdot,\, \mathbb{Z})$ of fibers in families of smooth curves of genus 1. In such families, the monodromy group acting on H^1 of the fiber is contained in $\mathrm{SL}(2,\, \mathbb{Z})$.

If $p\colon \mathcal{X} \to B$ is a smooth family of elliptic curves, then the morphism $B \to \mathbb{A}^1$ assigning the j-invariant $j(p^{-1}(b))$ to a point $b \in B$, will be denoted by $J_{\mathcal{X}}$. Following Miranda [13, Lecture V], we will say that $J_{\mathcal{X}}$ is *the J-map* of the family $\mathcal{X}$ (in Kodaira's paper [12], the morphism $J_{\mathcal{X}}$ is called *analytic invariant of the family $\mathcal{X}$*). If $b_0 \in B$, then the monodromy representation $\pi_1(B,\, b_0) \to \mathrm{SL}(H^1(\mathcal{X}_{b_0},\, \mathbb{Z}))$ will be denoted by $\rho_{\mathcal{X}}$.

Suppose now that the base B is smooth and connected. Since the fiber over the generic point of B is an elliptic curve over the field of rational functions $\mathbb{C}(B)$, and since this elliptic curve can be reduced to the Weierstrass normal from, there exists a Zariski open subset $U \subset B$ such that the restriction $\mathcal{X}|_U$ is isomorphic to the family

$$w^2 = z^3 + Pz + Q, \tag{2}$$

where P and Q are regular functions on U, the fiber over $b \in B$ being the smooth projective model of the curve defined by the equation $w^2 = z^3 + P(b)z + Q(b)$, and discriminant of the right-hand side of (2) does not vanish on U. Proposition 1.2 shows that $\operatorname{Mon}(\mathcal{X}) = \operatorname{Mon}(\mathcal{X}|_U)$, so, as far as monodromy groups are concerned, we may and will assume that $U = B$ and that the family is defined by (2) with non-vanishing discriminant.

Any such family of the form (2) defines a morphism $\operatorname{Br}_{\mathcal{X}}\colon B \to \mathbb{C}^{(3)}$ assigning to each point $b \in B$ the collection of roots of $z^3 + P(b)z + Q(b)$. If $b_0 \in B$ and if $\{u, v, w\}$ is the set of roots of the polynomial $z^3 + P(b)z + Q(b)$, then the morphism $\operatorname{Br}_{\mathcal{X}}$ induces a homomorphism $\operatorname{br}_{\mathcal{X}}\colon \pi_1(B, b_0) \to B_3(u, v, w)$. If $\mathcal{X}_{b_0}$ is the fiber of $\mathcal{X}$ over b_0, and if

$$\mu\colon B_3(u, v, w) \to \operatorname{SL}(H^1(X_{b_0}), \mathbb{Z})) = \operatorname{SL}(2, \mathbb{Z})$$

is the homomorphism defined in Section 2, then the diagram

$$\begin{array}{ccc} \pi_1(B, b_0) & \xrightarrow{\ \rho_{\mathcal{X}}\ } & \operatorname{SL}(H^1(X_{b_0}), \mathbb{Z})) \\ & {\scriptstyle \operatorname{br}_{\mathcal{X}}}\searrow \qquad \nearrow{\scriptstyle \mu} & \\ & B_3 & \end{array}$$

is commutative.

Suppose that $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ are two families of elliptic curves over a base B. One says that $\mathcal{X}_1$ and $\mathcal{X}_2$ *differ by a quadratic twist* if their scheme-theoretic generic fibers (which are elliptic curves over the field of rational functions $K = \mathbb{C}(B)$) are isomorphic over a quadratic extension of K, that is, there exists a morphism $B' \to B$ of degree 2 such that $(\mathcal{X}_1)_{B'}$ and $(\mathcal{X}_2)_{B'}$ are isomorphic smooth families. If the families $\mathcal{X}_1$ and $\mathcal{X}_2$ differ by a quadratic twist, then they can be represented by Weierstrass equations

$$\begin{aligned} \mathcal{X}_1\colon\ & y^2 = x^3 + Px + Q, \\ \mathcal{X}_2\colon\ & y^2 = x^3 + D^2 \cdot Px + D^3 \cdot Q, \end{aligned} \tag{3}$$

where D is a rational function on B (see [15, Chapter X, Proposition 5.4]).

Being interested only in the monodromy groups $\operatorname{Mon}(\mathcal{X}_1)$ and $\operatorname{Mon}(\mathcal{X}_2)$, we can, replacing B by a Zariski open subset if necessary, assume that the families $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth; in particular, this implies that D is a regular function on B without zeroes.

Suppose that B is a smooth algebraic variety, D is a regular function on B without zeroes, and $b_0 \in B$ is a point. In the definition that follows we regard B as a complex manifold and D as a holomorphic function on B.

Definition 3.1. In the above setting, by $\chi_D\colon \pi_1(B, b) \to \{\pm 1\}$ we denote the homomorphism defined as follows. If $B_0 \in B$, $\gamma \in \pi_1(B, b_0)$, we put $\chi_D(\gamma) = -1$ if the function $\sqrt{D}$ changes after the analytic continuation along a loop representing γ, and we put $\chi_D(\gamma) = 1$ otherwise. In other words, if a loop representing γ is of the form $t \mapsto \varphi(t)$, $t \in [0; 1]$, then $\chi_D(\gamma) = (-1)^k$, where k is the number of times the loop $D \circ \varphi$ winds around the origin.

We will say that χ_D is the *quadratic character associated to* D.

Proposition 3.2. *In the above setting, suppose that* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are smooth families of elliptic curves that differ by a quadratic twist as in* (3)*. Then the monodromy homomorphism* $\rho_{\mathcal{X}_2} \colon \pi_1(B) \to \mathrm{SL}(2, \mathbb{Z})$ *differs from* $\chi_D \rho_{\mathcal{X}_1}$ *by an inner automorphism of* $\mathrm{SL}(2, \mathbb{Z})$.

Proof. Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are defined by the equations (3), where D has no zeroes or poles on B and discriminants of the left-hand sides of never vanish. If $u(b)$, $v(b)$, and $w(b)$ are the roots of the polynomial $x^3+P(b)x+Q(b)$, where $b \in B$, then roots of the polynomial $x^3 + D(b)^2P(b)x + D(b)^3Q(b)$ are $D(b)u(b)$, $D(b)v(b)$, and $D(b)w(b)$.

Choose a base point $b_0 \in B$ and fix a path τ in $\mathbb{C}^{(3)}$ joining the points (unordered triples) $(D(b_0)u(b_0), D(b_0)v(b_0), D(b_0)w(b_0))$ and $(u(b_0), v(b_0), w(b_0))$. If $\gamma \in \pi_1(B, b_0)$, then

$$\mathrm{br}_{\mathcal{X}_2}(\gamma) = \tau\, \mathrm{br}_{\mathcal{X}_1}(\gamma)\delta\tau^{-1},$$

where $\delta \in B_3(u(b_0), v(b_0), w(b_0))$ is the loop defined by the formula

$$t \mapsto \frac{D(\gamma(t))}{|D(\gamma(t))|} \cdot \gamma(t)$$

(if $\lambda \in \mathbb{C}^*$ and $\alpha = (u, v, w) \in \mathbb{C}^{(3)}$, then $\lambda \cdot \alpha = (\lambda u, \lambda v, \lambda w)$).

If the loop δ winds k times around the origin, then Proposition 2.1 implies that $\mu(\delta) = (-1)^k I = \chi_D(\gamma) I$ (I is the identity matrix), whence the result. □

Lemma 3.3. *Suppose that* Π *is a group and that* $\rho \colon \Pi \to \mathrm{SL}(2, \mathbb{Z})$ *and* $\chi \colon \Pi \to \{\pm 1\}$ *are homomorphisms. Put* $G_1 = \mathrm{Im}\,\rho$, $G_2 = \mathrm{Im}(\chi\rho)$*. Then one of the following cases holds*:

(i) $G_1 = G_2$;
(ii) *there exists a subgroup* $H \subset G_1$, $(G_1 : H) = 2$, *such that* $G_2 = H \cup (-I)(G_1 \setminus H)$;
(iii) G_2 *is the subgroup of* $\mathrm{SL}(2, \mathbb{Z})$ *generated by* G_1 *and* $-I$.

Proof. If χ is trivial, then $G_2 = G_1$; otherwise $\mathrm{Ker}\,\chi$ is a subgroup of index 2 in Π.

If $\mathrm{Ker}\,\rho \subset \mathrm{Ker}\,\chi$, then the character χ factors through the group G_1 and $G_2 = H \cup (-I)(G_1 \setminus H)$, where H is the kernel of the induced homomorphism $G_1 \to \{\pm 1\}$.

If $\mathrm{Ker}\,\rho \not\subset \mathrm{Ker}\,\chi$, then $G_1 = \rho(\mathrm{Ker}\,\chi) = \rho(\Pi \setminus \mathrm{Ker}\,\chi)$, so G_2 is generated by G_1 and $-I$. □

Suppose now that $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth families of elliptic curves over the same base B. If we fix a base point $b \in B$, we can identify (not canonically) first integer cohomology groups of the fibers $(\mathcal{X}_1)_b$ and $(\mathcal{X}_2)_b$ and identify them both with $\mathbb{Z}^2$.

Proposition 3.4. *Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth families of elliptic curves over the same base B and that $\mathcal{X}_1$ and $\mathcal{X}_2$ differ by a quadratic twist with a non-vanishing regular function D as in (3). Put $G_i = \mathrm{Mon}(\mathcal{X}_i, b) \subset \mathrm{SL}(2, \mathbb{Z})$ (these subgroups are only defined up to a conjugation). Then:*

(i) *Either G_1 and G_2 are conjugate, or one of these groups contains a subgroup of index 2 that is conjugate to the other subgroup, or each G_i contains a subgroup H_i of index 2 and the subgroups H_1 and H_2 are conjugate.*
(ii) *If $G_1 = \mathrm{SL}(2, \mathbb{Z})$, then $G_2 = \mathrm{SL}(2, \mathbb{Z})$.*

Proof. Put $\Pi = \pi_1(B)$. Proposition 3.2 implies that, conjugating the subgroups if necessary, one may assume that there exist homomorphisms $\rho\colon \Pi \to \mathrm{SL}(2, \mathbb{Z})$ and $\chi\colon \Pi \to \{\pm 1\}$ such that $G_1 = \mathrm{Im}(\rho)$, $G_2 = \mathrm{Im}(\chi\rho)$.

Now part (i) follows immediately from Lemma 3.3.

To prove part (ii), one has only to account for case (ii) of Lemma 3.3. To that end observe that $\mathrm{SL}(2, \mathbb{Z})$ contains a unique subgroup H of index 2: this follows from the fact that the abelianization of $\mathrm{SL}(2, \mathbb{Z})$ is $\mathbb{Z}/12\mathbb{Z}$. Since the corresponding epimorphism $\varphi\colon \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{Z}/12\mathbb{Z}$ maps $-I$ to $6 \bmod 12$, one has $-I \in H$ and $G_2 = G_1 = \mathrm{SL}(2, \mathbb{Z})$. □

Corollary 3.5 (from the proof)**.** *Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are families of elliptic curves over the same smooth base that differ by a quadratic twist. Then*

(i) *if the monodromy group $\mathrm{Mon}(\mathcal{X}_1)$ has finite index in $\mathrm{SL}(2, \mathbb{Z})$, then either the indices $(\mathrm{SL}(2, \mathbb{Z}) : \mathrm{Mon}(\mathcal{X}_1))$ and $(\mathrm{SL}(2, \mathbb{Z}) : \mathrm{Mon}(\mathcal{X}_2))$ are equal or one of them is twice greater than the other*;
(ii) *the images of $\mathrm{Mon}(\mathcal{X}_1)$ and $\mathrm{Mon}(\mathcal{X}_2)$ in $\mathrm{PSL}(2, \mathbb{Z})$ are conjugate.* □

Proposition 3.6. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are smooth families of elliptic curves over the same base B and that their J-maps $J_\mathcal{X}, J_\mathcal{Y}\colon B \to \mathbb{A}^1$ are equal and non-constant. Then*

(i) *either $(\mathrm{SL}(2, \mathbb{Z}) : \mathrm{Mon}(\mathcal{X})) = (\mathrm{SL}(2, \mathbb{Z}) : \mathrm{Mon}(\mathcal{Y}))$ or one of these indices is twice greater than the other (we allow indices of subgroups to be infinite and assume that $2 \cdot \infty = \infty$)*;
(ii) *the images of $\mathrm{Mon}(\mathcal{X})$ and $\mathrm{Mon}(\mathcal{Y})$ in $\mathrm{PSL}(2, \mathbb{Z})$ are conjugate*;
(iii) *if $\mathrm{Mon}(\mathcal{X}) = \mathrm{SL}(2, \mathbb{Z})$ then $\mathrm{Mon}(\mathcal{Y}) = \mathrm{SL}(2, \mathbb{Z})$.*

Proof. The (scheme-theoretic) generic fibers of the families $\mathcal{X}$ and $\mathcal{Y}$ over $\mathrm{Spec}\, K$, where $K = \mathbb{C}(B)$ (the field of rational functions), have the same j-invariant $J_\mathcal{X} = J_\mathcal{Y} \in \mathbb{C}(B)$, and this j-invariant is not equal to 0 or 1728 since $J_\mathcal{X} = J_\mathcal{Y}$ is not constant. Hence, these elliptic curves differ by a quadratic twist (see [15, Chapter X, Proposition 5.4]), and so do the corresponding families. □

4. Main Result and Applications

We begin with a folklore result.

Proposition 4.1. *Suppose that $\pi\colon \mathcal{X} \to B$ is a smooth family of curves of genus 1, where B is an algebraic variety. Then the mapping from B to $\mathbb{C}$ that assigns j-invariant $j(f^{-1})(b)$ to a point $b \in B$, is induced by a morphism from B to $\mathbb{A}^1$.*

Proof. If π has a section, see [6, § 5]; the general case is treated by passing to the relative Picard. □

Proposition 4.2. *Suppose that $\pi\colon \mathcal{X} \to B$ is a smooth family of curves of genus 1 over a smooth and connected base B (the ground field is $\mathbb{C}$); let $J_{\mathcal{X}}\colon B \to \mathbb{A}^1$ be the J-map, attaching to any point $a \in B$ the j-invariant of the fiber of $\mathcal{X}$ over a.*

(i) *If the morphism $J_{\mathcal{X}}$ is not constant and its general fiber is connected, then* $\mathrm{Mon}(X) = \mathrm{SL}(2, \mathbb{Z})$.

(ii) *If the morphism $J_{\mathcal{X}}$ is not constant and its general fiber has $m \geqslant 2$ connected components, then* $\mathrm{Mon}(\mathcal{X})$ *is a subgroup of index at most $2m$ in* $\mathrm{SL}(2, \mathbb{Z})$ *and the image of* $\mathrm{Mon}(\mathcal{X})$ *in* $\mathrm{PSL}(2, \mathbb{Z})$ *is a subgroup of index at most m in* $\mathrm{PSL}(2, \mathbb{Z})$.

Proof. If the family $\pi\colon \mathcal{X} \to B$ does not have a section, replace it by the relative Picard $\pi'\colon \mathcal{X}' = \mathrm{Pic}^0(\mathcal{X}/B) \to B$, which will not affect the J-map or the index $(\mathrm{SL}(2, \mathbb{Z}) : \mathrm{Mon}(\mathcal{X}))$. Assuming now that $\mathcal{X}$ has a section, put

$$V = \{(p, q) \in \mathbb{A}^2 \colon 4p^3 + 27q^2 \neq 0\}$$

and consider the smooth family of elliptic curves $\mathcal{B} \to V$ in which the fiber over (p, q) is the smooth projective model $C_{p,q}$ of the curve with equation $y^2 = x^3+px+q$ and the section assigns to (p, q) the "point at infinity" of this model. It is well known (see for example [1, Corollary to Theorem 1]) that $\mathrm{Mon}(\mathcal{B}) = \mathrm{SL}(2, \mathbb{Z})$.

Now put $\mathbb{A}^1_0 = \mathbb{A}^1 \setminus \{0\}$, $\mathbb{A}^1_{0,1728} = \mathbb{A}^1 \setminus \{0, 1728\}$ and

$$V' = J_{\mathcal{B}}^{-1}(\mathbb{A}^1_{0,1728}) = \{(p, q) \in \mathbb{A}^2 \colon 4p^3 + 27q^2 \neq 0,\ \ p \neq 0,\, q \neq 0\}.$$

Let $\mathcal{B}'$ be the restriction of the family $\mathcal{B}$ to V'; put $B' = J_{\mathcal{X}}^{-1}(\mathbb{A}^1_{0,1728})$, and let $\mathcal{X}'$ be the restriction of $\mathcal{X}$ to B'. Proposition 1.2 implies that $\mathrm{Mon}(\mathcal{X}') = \mathrm{Mon}(\mathcal{X})$ and $\mathrm{Mon}(\mathcal{B}') = \mathrm{Mon}(\mathcal{B}) = \mathrm{SL}(2, \mathbb{Z})$.

Observe that there exists an isomorphism $g\colon V \to \mathbb{A}^1_0 \times \mathbb{A}^1_{0,1728}$ such that the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\ g\ } & \mathbb{A}^1_0 \times \mathbb{A}^1_{0,1728} \\ & {\scriptstyle J_{\mathcal{X}'}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & \mathbb{A}^1_{0,1728} & \end{array}$$

is commutative. Indeed, one can define g by the formula $(p, q) \mapsto (q/p,\, j(C_{p,q}))$, and the inverse morphism will be

$$(\lambda, j) \mapsto \left(\frac{\lambda^2}{\frac{4}{27}\left(\frac{1728}{j} - 1\right)},\ \frac{\lambda^3}{\frac{4}{27}\left(\frac{1728}{j} - 1\right)}\right).$$

Hence, in the fibered product

$$\begin{array}{ccc} W & \xrightarrow{\ f\ } & V' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle J_{\mathcal{B}'}} \\ B' & \xrightarrow[J_{\mathcal{X}'}]{} & \mathbb{A}^1_{0,1728} \end{array}$$

the variety W is isomorphic to $(\mathbb{A}^1_0) \times B'$ (in particular, W is smooth and irreducible) and fibers of f are isomorphic to fibers of $J_{\mathcal{X}_0}$. Thus, the hypothesis implies that a general fiber of the morphism $J_{\mathcal{X}_0}$ has m connected components. On the other hand, any fiber of the morphism u is irreducible since it is isomorphic to $\mathbb{A}^1_0$. Now Proposition 1.3 implies that for the pullback families $\mathcal{B}'_W$ and $\mathcal{X}'_W$ on W, the group $\operatorname{Mon}(\mathcal{B}'_W)$ is a subgroup of index at most m in $\operatorname{Mon}(\mathcal{B}) = \mathrm{SL}(2,\,\mathbb{Z})$ and $\operatorname{Mon}((\mathcal{X}_0)_W) = \operatorname{Mon}(\mathcal{X}_0)$ (up to a conjugation).

Since $J_{\mathcal{B}'_W} = J_{\mathcal{B}'} \circ f = J_{\mathcal{X}'} \circ u = J_{\mathcal{X}'_W}$, Proposition 3.6 implies the result. $\square$

Remark 4.3. I do not know whether the bound in this proposition can be improved to $(\mathrm{PSL}(2,\,\mathbb{Z}) : \operatorname{Mon}(\mathcal{X})) \leqslant m$ for $m > 1$.

Corollary 4.4. *If $\mathcal{X} \to B$ is a non-isotrivial smooth family of curves of genus 1, then its monodromy group is a subgroup of finite index in $\mathrm{SL}(2,\,\mathbb{Z})$.*

Here is the first application of what we proved.

Proposition 4.5. *If B is a smooth algebraic variety with abelian fundamental group, then any smooth family $\pi\colon \mathcal{X} \to B$ of curves of genus 1 must be isotrivial.*

Proof. Suppose that $\pi_1(B) = G$ is abelian. If the J-map $J_{\mathcal{X}}\colon B \to \mathbb{A}^1$ is not constant, then Corollary 4.4 asserts that the group $\operatorname{Mon}(\mathcal{X})$ has finite index in $\mathrm{SL}(2,\,\mathbb{Z})$. Since $\Gamma(2)$ has finite index in $\mathrm{SL}(2,\,\mathbb{Z})$, one has $(\Gamma(2) : \Gamma(2) \cap \operatorname{Mon}(\mathcal{X})) < \infty$. If G is the image of $\Gamma(2) \cap \operatorname{Mon}(\mathcal{X})$ in $\Gamma(2)/\{\pm I\}$, then G is an abelian subgroup of finite index in $\Gamma(2)/\{\pm I\}$, which is impossible since the latter is isomorphic to the free group with two generators. $\square$

For the case of non-commutative π_1 of the base, one can obtain an upper bound on the index of monodromy groups in non-isotrivial families.

Proposition 4.6. *Suppose that $\mathcal{X} \to B$ is a smooth non-isotrivial family of curves of genus 1 over a smooth base B and that $\pi_1(B)$ can be generated by $r \geqslant 2$ elements. Then $(\mathrm{SL}(2,\,\mathbb{Z}) : \operatorname{Mon}(\mathcal{X})) \leqslant 12(r-1)$.*

Corollary 4.7. *If $\mathcal{X} \to C$ is a non-isotrivial family of elliptic curves over a smooth curve of genus g, with s degenerate fibers, then $(\mathrm{SL}(2,\,\mathbb{Z}) : \operatorname{Mon}(\mathcal{X})) \leqslant 12(2g+s-1)$.*

Proposition 4.6 is a consequence of the following elementary lemma.

Lemma 4.8. *Suppose that $G \subset \mathrm{SL}(2,\,\mathbb{Z})$ is a subgroup of finite index and that G can be generated by r elements. Then $(\mathrm{SL}(2,\,\mathbb{Z}) : G) \leqslant 12(r-1)$.*

Proof of the lemma. Throughout the proof, the free group with m generators will be denoted by F_m.

Since G can be generated by r elements, there exists an epimorphism $\pi\colon F_r \twoheadrightarrow G$. Putting $H = p^{-1}(G \cap \Gamma(2))$, one obtains the following commutative diagram of embeddings and surjections:

$$\begin{array}{ccccccc}
H & \overset{p'}{\twoheadrightarrow} & G \cap \Gamma(2) & \hookrightarrow & \Gamma(2) & & \\
\downarrow{\scriptstyle \text{index}=d} & & \downarrow{\scriptstyle \text{index}=d} & & \downarrow{\scriptstyle \text{index}=6} & & \\
F_r & \underset{p}{\twoheadrightarrow} & G & \underset{i}{\hookrightarrow} & \mathrm{SL}(2,\,\mathbb{Z}) & \underset{j}{\twoheadrightarrow} & \mathrm{SL}(2,\,\mathbb{Z})/\Gamma(2) \cong S_3.
\end{array} \qquad (4)$$

If $d \leqslant 6$ is the order of $\mathrm{Im}(j \circ i)$, then $(G : G \cap \Gamma(2)) = (F_r : H) = d$, so by Schreier's theorem $H \cong F_{d(r-1)+1}$. Since the morphism p' is surjective, the group $G \cap \Gamma(2)$ can be generated by $d(r-1)+1$ elements. Put $F = \Gamma(2)/\{\pm I\}$, and let $\pi\colon \Gamma(2) \to F$ be the natural projection. The subgroup $\pi(G \cap \Gamma(2)) \subset F$ can be also generated by $d(r-1)+1$ elements; since $F \cong F_2$, Schreier's theorem implies that $\pi(G \cap \Gamma(2)) \cong F_m$, where $m \leqslant d(r-1)+1$. Applying Schreier's for the third time, we obtain that $(F : \pi(H \cap \Gamma(2))) = m - 1 \leqslant d(r-1)$, whence $(\Gamma(2) : G \cap \Gamma(2)) \leqslant 2(m-1) \leqslant 2d(r-1)$. It follows from the right-hand square of the diagram (4) that

$$(\mathrm{SL}(2,\,\mathbb{Z}) : G) = \frac{(\mathrm{SL}(2,\,\mathbb{Z}) : \Gamma(2)) \cdot (\Gamma(2) : G \cap \Gamma(2))}{(G : G \cap \Gamma(2))} \leqslant \frac{12d(r-1)}{d},$$

whence the result. □

Proof of Proposition 4.6. Put $\mathrm{Mon}(\mathcal{X}) = G \subset \mathrm{SL}(2,\,\mathbb{Z})$. Since $\pi_1(B)$ can be generated by r elements, the same is true for G; now Corollary 4.4 implies that $(\mathrm{SL}(2,\,\mathbb{Z}) : G) < +\infty$, and Lemma 4.8 applies. □

Using Proposition 4.2 one can obtain other lower bounds for monodromy groups. Observe first that the named proposition immediately implies the following corollary.

Corollary 4.9. *If $\pi\colon \mathcal{X} \to C$ is a family of curves of genus 1 over a smooth projective curve C and if $J_{\mathcal{X}}\colon C \dashrightarrow \mathbb{A}^1$ is its J-map, and if $J_{\mathcal{X}}$ is not constant, then $(\mathrm{SL}(2,\,\mathbb{Z}) : \mathrm{Mon}(\mathcal{X})) \leqslant 2 \deg J_{\mathcal{X}}$.*

If $\mathcal{X}$ is smooth and π has a section, one can be more specific.

Proposition 4.10. *Suppose that $\pi\colon \mathcal{X} \to C$ is a minimal smooth elliptic surface with section (it means that $\mathcal{X}$ is a smooth projective surface, C is a smooth projective curve, the general fiber of π is a smooth curve of genus 1, no fiber of π contains a rational (-1)-curve, and p has a section) and that $J_{\mathcal{X}}$ is not constant.*
Then

$$(\mathrm{SL}(2,\,\mathbb{Z}) : \mathrm{Mon}(\mathcal{X})) \leqslant 2 \cdot \sum_{s \in C} e(s), \tag{5}$$

where $e(s) = n$ if the fiber over s is a cycle of n smooth rational curves or the nodal rational curve if $n = 1$ (type I_n in Kodaira's classification [12], [13]), $e(s) = n$ if the fiber over s consists of $n + 5$ smooth rational curves with intersection graph isomorphic to the extended Dynkin graph $\tilde{D}_{n+4}$, $n \geqslant 1$ (type I_n^* in Kodaira's classification), and $e(s) = 0$ otherwise.

Proof. In view of Corollary 4.9 the index in the left-hand side of (5) is less or equal to $2 \deg J_{\mathcal{X}}$, and $\deg J_{\mathcal{X}}$ equals $\sum_{s \in C} e(s)$ by virtue of Corollary IV.4.2 from [13]. □

Similarly, one can express $\deg J_{\mathcal{X}}$ (and obtain a lower bound for $\mathrm{Mon}(\mathcal{X})$) using the information about the points where j-invariant of the fiber (smooth or not) equals 0 or 1728, see for example [13, Lemma IV.4.5, Table IV.3.1] (in the notation of [13], j-invariant is 1728 times less than that defined by (1)).

5. A Remark on Families of Hyperelliptic Curves

Proposition 5.1. *If $\pi\colon \mathcal{X} \to B$ is a smooth family of hyperelliptic curves of genus $g > 2$, then*

$$(\mathrm{Sp}(2g,\,\mathbb{Z}) : \mathrm{Mon}(\mathcal{X})) \geqslant \frac{2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdots(2^2-1)}{(2g+2)!}.$$

Corollary 5.2. *If $\pi\colon \mathcal{X} \to B$ is a smooth family of hyperelliptic curves of genus $g > 2$, then $\mathrm{Mon}(\mathcal{X})$ is a proper subgroup of $\mathrm{Sp}(2g,\,\mathbb{Z})$.*

Proof of Proposition 5.1. In this proof, $\mathrm{Mon}(\mathcal{X},\,\mathbb{Z})$ will denote the monodromy group acting on the integer H^1 of a fiber of $\mathcal{X}$, and $\mathrm{Mon}(\mathcal{X},\,\mathbb{Z}/2\mathbb{Z})$ will stand for the monodromy group acting on $H^1(\text{fiber},\,\mathbb{Z}/2\mathbb{Z})$.

The natural surjection $\mathrm{Sp}(2g,\,\mathbb{Z}) \to \mathrm{Sp}(2g,\,\mathbb{Z}/2\mathbb{Z})$ implies that

$$(\mathrm{Sp}(2g,\,\mathbb{Z}) : \mathrm{Mon}(\mathcal{X},\,\mathbb{Z})) \geqslant (\mathrm{Sp}(2g,\,\mathbb{Z}/2\mathbb{Z}) : \mathrm{Mon}(\mathcal{X},\,\mathbb{Z}/2\mathbb{Z})),$$

so it suffices to show that

$$(\mathrm{Sp}(2g,\,\mathbb{Z}/2\mathbb{Z}) : \mathrm{Mon}(\mathcal{X},\,\mathbb{Z}/2\mathbb{Z})) \geqslant \frac{2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdots(2^2-1)}{(2g+2)!}. \qquad (6)$$

To that end, let X be a hyperelliptic curve of genus $g \geqslant 2$ that is a fiber of $\mathcal{X}$; denote its Weierstrass points by $P_1,\,\dots,\,P_{2g+2}$. It is well known (see for example [5, Lemma 2.1]) that the 2-torsion subgroup $(\mathrm{Pic}(X))_2 \subset \mathrm{Pic}(X)$ is generated by classes of divisors $P_i - P_j$. Since $\mathrm{Pic}(X)_2 \cong H^1(X,\,\mathbb{Z}/2\mathbb{Z})$, the action of $\pi_1(B_{\mathrm{sm}})$ on $H^1(X,\,\mathbb{Z}/2\mathbb{Z})$ is completely determined by the permutations of the Weierstrass points $P_1,\,\dots,\,P_{2g+2}$ it induces. Thus, order of $\mathrm{Mon}(\mathcal{X},\,\mathbb{Z}/2\mathbb{Z})$ is at most $(2g+2)!$. Since

$$(\mathrm{Sp}(2g,\,\mathbb{Z}/2\mathbb{Z}) : 1) = 2^{g^2}(2^{2g}-1)(2^{2(g-1)}-1)\cdots(2^2-1),$$

the proposition follows. □

Remark 5.3. The bound in Proposition 5.1 is sharp, which follows from A'Campo's paper [1]. To wit, if B is the space of polynomials $P(x) = x^{2g+2} + a_{2g}x^{2g} + \cdots + a_1x + a_0$ without multiple roots (here, $g \geqslant 2$), and if $\mathcal{X} \to B$ is the family in which the fiber over P is the smooth projective model of the curve with equation $y^2 = P(x)$ (which is hyperelliptic of genus g), then the corollary on page 319 of [1] contains the assertion that the index $(\mathrm{Sp}(2g,\,\mathbb{Z}) : \mathrm{Mon}(\mathcal{X}))$ is equal to the right-hand side of (6). A description of the group $\mathrm{Mon}(\mathcal{X})$ can be found in the appendix to [3], which (the appendix) is devoted to the exposition of results of A.Varchenko.

6. Example: An Application to Del Pezzo Surfaces

Not much is known about hyperplane monodromy groups of (embedded) projective varieties, even surfaces. The classification of surfaces for which this group is trivial was found by Zak [17]: it turned out that surfaces with this property are either ruled or have hyperplane sections of genus 0. In this section we treat the case of surfaces for which the genus of hyperplane sections is 1.

Proposition 6.1. *If $X \subset \mathbb{P}^n$ is a Del Pezzo surface embedded by (a subsystem of) the anticanonical linear system $|-K_X|$, then the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of its smooth hyperplane sections is the entire* $\mathrm{SL}(2, \mathbb{Z})$.

First recall some notation and definitions.

If X is an algebraic variety and $\mathcal{R}$ is a coherent sheaf of reduced $\mathcal{O}_X$-algebras, we denote its relative spectrum (which is a scheme over X) by $\mathbf{Spec}\,\mathcal{R}$ (under our assumptions $\mathbf{Spec}\,\mathcal{R}$ is an algebraic variety and the canonical morphism $\mathbf{Spec}\,\mathcal{R} \to X$ is finite).

If $p \in \mathbb{P}^n$ is a point and $L \subset \mathbb{P}^n$ is a linear subspace, then $\overline{p, L}$ denotes the linear span of $\{p\} \cup L$.

If $A_1, \ldots, A_4$ are points on the affine line with coordinates $a_1, \ldots, a_4$, then by their cross-ratio we mean

$$[A_1, A_2, A_3, A_4] = \frac{a_3 - a_1}{a_3 - a_2} \Big/ \frac{a_4 - a_1}{a_4 - a_2}.$$

If $X \subset \mathbb{P}^n$ is a smooth projective variety and $X^* \subset (\mathbb{P}^n)^*$ is its projective dual, one can define the "universal smooth hyperplane section of X", that is, the family

$$\mathcal{U}_X = \{(x, \alpha) \in X \times ((\mathbb{P}^n)^* \setminus X^*)\colon x \in H_\alpha\}, \tag{7}$$

where $H_\alpha \subset \mathbb{P}^n$ is the hyperplane corresponding to the point $\alpha \in (\mathbb{P}^n)^*$. The morphism $\pi\colon (x, \alpha) \mapsto \alpha$ makes $\mathcal{X}$ a smooth family of n-dimensional projective varieties over $(\mathbb{P}^n)^* \setminus X^*$; for any natural d, this family induces a monodromy action of $\pi_1((\mathbb{P}^n)^* \setminus X^*)$ on $H^d(Y, \mathbb{Z})$, where Y is a smooth hyperplane section of X.

In the above setting, the image of $\pi_1((\mathbb{P}^n)^* \setminus X^*)$ in the group $\mathrm{Aut}(H^n(Y, \mathbb{Z}))$ will be called *hyperplane monodromy group* of X.

Lemma 6.2. *Suppose that $X \subset \mathbb{P}^n$ is a smooth projective variety and $p \in \mathbb{P}^n \setminus X$ is a point such that the projection with center p induces an isomorphism $\pi_p\colon X \to X' \subset \mathbb{P}^{n-1}$. If $H \ni p$ is a hyperplane that is transversal to X, then, after identifying $Y = X \cap H$ with $Y' = \pi_p(Y) = X' \cap \pi_p(H)$, the hyperplane monodromy groups acting on $H^*(Y, \mathbb{Z})$ and $H^*(Y', \mathbb{Z})$ are the same.*

The proof that is sketched below was suggested to me by Jason Starr.

Sketch of proof. Denote by $H_p \subset (\mathbb{P}^n)^*$ the hyperplane corresponding to the point $p \in \mathbb{P}^n$. It is clear that H_p is naturally isomorphic to $(\mathbb{P}^{n-1})^*$ and that $(X')^* = X^* \cap H_p$. Moreover, the hyperplane H_p is transversal to X^* at any smooth point of X^* (indeed, if H_p is tangent to X^* at a smooth point, then $p \in (X^*)^* = X$, which contradicts the hypothesis).

To prove the lemma it suffices to show that the natural mapping $\pi_1(H_p \setminus (X')^*) \to \pi_1((\mathbb{P}^n)^* \setminus X^*)$ is a surjection. To that end observe that there exists a line $\ell \subset H_p$ that is transversal to the smooth part of $X^* \cap H_p = (X')^*$ and does not pass through singular points of $(X')^*$ (*a fortiori*, ℓ is transversal to X^*, too). Now $\pi_1(\ell \setminus X^*)$ surjects both onto $\pi_1(H_p \setminus (X')^*)$ and onto $\pi_1((\mathbb{P}^n)^* \setminus X^*)$, whence the desired surjectivity. □

Lemma 6.2 implies that when studying hyperplane monodromy groups one may assume that the variety in question is embedded by a complete linear system. Recall that if a Del Pezzo surface $X \subset \mathbb{P}^n$ is embedded by the complete linear system $|-K_X|$ then $\deg X = n \leqslant 9$; besides, if $n > 3$, $p \in X$ is a general point, and $\bar{X}$ is the blow-up of X at p, then the projection $\pi_p \colon X \dashrightarrow \mathbb{P}^{n-1}$ induces an isomorphism $\bar{\pi}_p \colon \bar{X} \to X' = \overline{\pi_p(X)} \subset \mathbb{P}^{n-1}$ and $X' \subset \mathbb{P}^{n-1}$ is a Del Pezzo surface embedded by $|-K_{X'}|$.

Lemma 6.3. *In the above setting, suppose that the hyperplane monodromy group of X' is the entire* $\mathrm{SL}(2, \mathbb{Z})$*. Then the hyperplane monodromy group of X is the entire* $\mathrm{SL}(2, \mathbb{Z})$ *as well.*

Sketch of proof. Observe that if a hyperplane $H \ni p$ is transversal to X, then $H \cap X$ is isomorphic to $H' \cap X'$, where $H' = \pi_p(H) \subset \mathbb{P}^{n-1}$, so each smooth hyperplane section of X' is a projection of hyperplane section of X. Now if variation of hyperplanes transversal to X *and* passing through p produces the entire group $\mathrm{SL}(2, \mathbb{Z})$, then this is the case for all hyperplanes transversal to X. A formal argument is left to the reader. □

Projecting Del Pezzo surfaces in $\mathbb{P}^n$, $n > 3$, consecutively from general points on them, one arrives at a cubic in $\mathbb{P}^3$; Lemma 6.3 implies that it suffices to prove Proposition 6.1 for this surface.

Suppose that $X \subset \mathbb{P}^3$ is a smooth cubic and $p \in X$ is a general point. Let $\bar{X}$ be the blow-up of X at p. The projection $\pi_p \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ induces a finite morphism $\bar{\pi}_p \colon \bar{X} \to \mathbb{P}^2$ of degree 2; the branch locus of this morphism is a smooth curve $C \subset \mathbb{P}^2$ of degree 4. For $\alpha \in (\mathbb{P}^2)^*$, denote the corresponding line by $\ell_\alpha \subset \mathbb{P}^2$. If ℓ_α is transversal to C (i.e., $\alpha \notin C^*$), then $\bar{\pi}_p^{-1}(\ell_\alpha)$ is smooth, irreducible, and isomorphic to $X \cap \overline{p, \ell_\alpha}$.

The proof of the following lemma is similar to that of Lemma 6.3.

Lemma 6.4. *Put*

$$\mathcal{X} = \{(\alpha, x) \in ((\mathbb{P}^2)^* \setminus C^*) \times \bar{X} \colon \bar{\pi}_p(x) \in \ell_\alpha\} \tag{8}$$

and denote the morphism $(\alpha, x) \mapsto \alpha$ *by* $\mathcal{X} \to (\mathbb{P}^2)^* \setminus C^*$*. If* $\mathrm{Mon}(\mathcal{X}, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$*, then the hyperplane monodromy group of X is also equal to* $\mathrm{SL}(2, \mathbb{Z})$*.*

Our next lemma is valid over any algebraically closed field.

Lemma 6.5. *Suppose that W is a smooth irreducible variety of dimension n, L is a smooth irreducible curve* (*we do not assume that W or L is projective*), *and $\varphi \colon W \to L$ is a proper and surjective morphism with* $(n-1)$*-dimensional fibers. Let $W \to Z \xrightarrow{v} L$ be the Stein factorization of φ.*

If there exists a point $p \in L$ such that $\varphi^{-1}(p)$ is irreducible and the morphism φ has maximal rank at a general point of $\varphi^{-1}(p)$, then $v \colon Z \to L$ is an isomorphism.

Proof. It is clear that Z is an irreducible and reduced curve. Since φ is proper and $\varphi^{-1}(p)$ is connected, the stalk $(\varphi_* \mathcal{O}_W)_p$ is a local ring, so $v^{-1}(p)$ consists of one point; denote this point by z. I claim that that z is a smooth point of Z and the morphism v is unramified at z. Indeed, let $\tau \in \mathcal{O}_{L,p}$ be a generator of the maximal ideal. Its image $v^*\tau \in \mathcal{O}_{Z,z}$ can be represented by a regular function $f \in \mathcal{O}_W(\varphi^{-1}(U))$, where $U \subset L$ is a Zariski neighborhood of p. Since the morphism φ has maximal rank at a general point of $\varphi^{-1}(p)$, the function $v^*\tau$ vanishes on the irreducible divisor $\varphi^{-1}(p)$ with multiplicity 1. Since regular functions on $\varphi^{-1}(U)$ must be constant on the fibers of the proper morphism φ, any element of the maximal ideal of the local ring $\mathcal{O}_{Z,z}$ is representable by a regular function $g \in \mathcal{O}_W(\varphi^{-1}(V))$, where V is a Zariski neighborhood of p, such that the zero locus of g in $\varphi^{-1}(V)$ coincides with $u^{-1}(z)$. Hence, $v^*\tau$ generates the maximal ideal of $\mathcal{O}_{Z,z}$, which proves our claim.

Since $v^{-1}(p) = \{z\}$, Z is smooth at z, and v is unramified at z, we conclude that the finite morphism v has degree 1. Since L is smooth, Zariski main theorem implies that v is an isomorphism. □

Proposition 6.6. *Suppose that* $\pi\colon X \to \mathbb{P}^2$ *is a finite morphism of degree* 2 *branched over a smooth quartic* $C \subset \mathbb{P}^2$*, where* X *is smooth. If* $J\colon (\mathbb{P}^2)^* \setminus C^* \to \mathbb{A}^1$ *is the morphism* $\alpha \mapsto j(\pi^{-1}(\ell_\alpha))$*, where* ℓ_α *is the line in* $\mathbb{P}^2$ *corresponding to* $\alpha \in (\mathbb{P}^2)^*$*, then a general fiber of* J *is irreducible.*

Proof. Let us show that the morphism J extends to a morphism

$$J_1\colon (\mathbb{P}^2)^* \setminus (C^*)_{\mathrm{sing}} \to \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}.$$

Indeed, if $\ell \subset \mathbb{P}^2$ is a line and $\ell \cap C = \{P_1, P_2, P_3, P_4\}$, then the curve $\pi^{-1}(\ell)$ is a curve of genus 1 and

$$j(\pi^{-1}(\ell)) = 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}, \tag{9}$$

where λ is the cross-ratio $[P_1, P_2, P_3, P_4]$, in no matter what order (see for example [15, Chapter III, Proposition 1.7b]). If α is a smooth point of $C^* \subset (\mathbb{P}^2)^*$, then the line ℓ_α is tangent to C at exactly one point that is not an inflection point. Thus, as the line ℓ tends to ℓ_α, exactly two intersection points from $\ell \cap C$ merge, so the cross-ratio of these four points tends to 0 (or 1, or ∞, depending on the ordering), and formula (9) shows that $j(\pi^{-1}(\ell))$ tends to ∞. This proves the existence of the desired extension.

Our argument shows that $J_1^{-1}(\infty) = C^* \setminus (C^*)_{\mathrm{sing}}$; if we regard J_1 as a rational mapping from $(\mathbb{P}^2)^*$ to $\mathbb{P}^1$ and if

$$\begin{array}{ccc} W & & \\ \downarrow{\scriptstyle\sigma} & \searrow^{J_2} & \\ (\mathbb{P}^2)^* & \overset{J_1}{\dashrightarrow} & \mathbb{P}^1 \end{array} \tag{10}$$

is a minimal resolution of indeterminacy for J_1, then $J_2^{-1}(\infty)$ equals the strict transform of C^* with respect to σ.

Now I claim that, at a general point of $J_2^{-1}(\infty)$, the derivative of J_2 has rank 1. It suffices to prove this assertion for J_1 and a general smooth point of C^*. To that end it suffices to construct an analytic mapping $\gamma\colon D \to (\mathbb{P}^2)^*$, where D is a disk in the complex plane with center at 0, such that $\gamma(D \setminus \{0\}) \subset (\mathbb{P}^2)^* \setminus C^*$, $\gamma(0)$ is a smooth point of C^*, and $|j(\pi^{-1}(\ell_{\gamma(t)}))| \sim \mathrm{const}/|t|$.

Suppose that a point $c \in C$ is not an inflection point nor a tangency point of a bitangent; if $\ell_\alpha \subset \mathbb{P}^2$ is the tangent line to C at c, then α s a smooth point of C^*. Now choose affine (x, y)-coordinates in $\mathbb{P}^2$ so that $c = (0, 0)$, the tangent ℓ_α has equation $y = 0$, and $\ell_\alpha \cap C = \{c, (C, 0), (D, 0)\}$, where $C, D \neq 0$ (so the remaining two points of $\ell_\alpha \cap C$ are in the finite part of $\mathbb{P}^2$ with respect to the chosen coordinate system). If $\ell_{\gamma(t)}$ is the line with affine equation $y = t$, then, for all small enough t, one has $\ell_{\gamma(t)} \cap C = \{A(t), B(t), C(t), D(t)\}$, where the x-coordinates of $A(t)$ and $B(t)$ are $\sqrt{t} + o(\sqrt{|t|})$ (for both values of $\sqrt{t}$), while the x coordinates of $C(t)$ and $D(t)$ tend to finite and non-zero numbers C and D. Hence,

$$\bigl|[C(t), A(t), B(t), D(t)]\bigr| \sim \frac{\text{const}}{\sqrt{|t|}} \quad \text{as } t \to 0;$$

formula (9) implies that $|j(\pi^{-1}(\ell_t))| \sim \text{const}/|t|$, as desired.

Let

$$\begin{array}{ccc} W & \xrightarrow{\;J_2\;} & \mathbb{P}^1 \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle v} & \\ & Z & \end{array}$$

be the Stein factorization in which W is a blow-up of $(\mathbb{P}^2)^*$ (see (10)), $Z = \mathbf{Spec}(J_2)_*\mathcal{O}_W$, and v is a finite morphism. Applying Lemma 6.5 with $L = \mathbb{P}^1$, $\varphi = J_2$, and $p = \infty$, we conclude that v is an isomorphism. Thus, fibers of J_2 coincide with fibers of u; since the latter are connected, fibers of J_2 are connected as well. Bertini theorem implies that a general fiber of J_2 is smooth; since it is connected, it must be irreducible. This implies that a general fiber of J is irreducible. □

Proof of Proposition 6.1. In view of Proposition 6.2 and Lemmas 6.3 and 6.4, it suffices to prove that $\mathrm{Mon}(\mathcal{X}) = \mathrm{SL}(2, \mathbb{Z})$, where $\mathcal{X}$ is the family defined by (8). Proposition 6.6 shows that the family $\mathcal{X}$ defined by formula (8) satisfies the hypothesis of Proposition 4.2(i), whence $\mathrm{Mon}(\mathcal{X}) = \mathrm{SL}(2, \mathbb{Z})$. □

Remark 6.7. Our argument shows as well that if X is a Del Pezzo surface of degree 2, then the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of non-singular elements of the anticanonical linear system $|-K_X|$, is $\mathrm{SL}(2, \mathbb{Z})$.

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National Research University Higher School of Economics, Russian Federation

Federal Scientific Centre Science Research Institute of System Analysis at Russian Academy of Science (FNP FSC SRISA RAS)

E-mail address: lvovski@gmail.com